1. Consider the following differential equation

$$t^2y'' - ty' = 0. \tag{1}$$

(a) (5%) Verify that $y \equiv 0$ is a solution of (1).
(b) (10%) Find a non-trivial solution of (1) satisfying $y(0) = y'(0) = 0$.
(c) (5%) Do the results in (a) and (b) violate the Uniqueness Theorem?

2. (15%) Show that all solutions $x(t), y(t)$ of

$$\begin{cases} \dot{x} = y(e^x - 1) \\ \dot{y} = x + e^y \end{cases}$$

which start in the right half plane must remain there for all time.

3. (20%) (a) Consider the system of differential equations

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2). \end{cases}$$

Show that the equilibrium $(0, 0)$ is asymptotically stable. Can you prove this result by the linearization technique?
(b) Consider the system of differential equations

$$\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2). \end{cases}$$

Describe the behavior of the equilibrium $(0, 0)$ for $t > 0$.

4. (25%) Consider the forced vibration equation

$$m\ddot{u} + b\dot{u} + ku = \cos\omega t, \tag{2}$$

where $m > 0, k > 0$, and $b \geq 0$. Denote $\omega_0 = \sqrt{k/m}$ the natural frequency.
(a) Find the general solution of (2) when $\omega \neq \omega_0$.
(b) Assume $b = 0$. Describe the behavior of the solution when $\omega = \omega_0$.
(c) Now if we assume $b > 0$, show that all solutions tends to zero as $t \to \infty$.

5. (20%) Consider the second order differential equation $Y'' = AY$, where A is a constant matrix.
(a) Find the general solution if $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.
(b) Find the general solution if $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.