1. (10%) Let $X_1, \cdots, X_n$ be a random sample from a population with probability density function $f(x|\theta) = \frac{1}{\theta} 1_{(1<x<\theta)}$. Derive the probability density function of $X_{(1)} = \min\{X_1, \cdots, X_n\}$.

2. Let $X_1, \cdots, X_n$ be a random sample from $N(\theta, \theta^2)$ with $\theta > 0$.

(2a) (10%) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that $(\bar{X}, S^2)$ is a sufficient statistic for θ but it is not a complete sufficient statistic.

(2b) (10%) Let $W(X_1, \cdots, X_n) = X_1 + X_1^2$. Show that $E[W(X_1, \cdots, X_n)|(\bar{X}, S^2)]$ is an unbiased estimator of $(\theta + \theta^2)$, and $var(E[W(X_1, \cdots, X_n)|(\bar{X}, S^2)]) \le var(W(X_1, \cdots, X_n))$.

3. Let $X_1, \cdots, X_n$ be a random sample from $Bernoulli(p)$ with $n \ge 2$ and $p > 0$.

(3a) (5%) Find the maximum likelihood estimator, say, $\hat{\tau}$, of $\tau(p) = p(1-p)$ and show that $\hat{\tau}$ is not an unbiased estimator of $\tau(p)$.

(3b) (10%) Find the uniformly minimum variance unbiased estimator say, $\tilde{\tau}$, of τ and show that its variance cannot attain the the Cramér-Rao lower bound.

(3c) (10%) Find the asymptotic distributions of $\hat{\tau}$ for $p = 1/2$ and $p \ne 1/2$.

4. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ with σ^2 being a known positive value.

(4a) (5%) Find a uniformly most powerful size α test of $H_0 : \mu \ge \mu_0$ versus $H_A : \mu < \mu_0$, where μ_0 is a known constant.

(4b) (15%) Find an expression for the power function, say, $\beta(\mu)$ of the test in (4a), and show that $\beta(\mu) \ge \alpha$ for $\mu < \mu_0$.

(4c) (10%) Show that there does not exist a uniformly most powerful size α test of $H_0 : \mu = \mu_0$ versus $H_A : \mu \ne \mu_0$.

5. Consider the simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, where ε_i's are independent and identically distributed with the normal distribution $N(0, \sigma^2)$, $i = 1, \cdots, n$.

(5a) (5%) Compute the maximum likelihood estimators of β_0, β_1, and σ^2.

(5b) (10%) Let $\hat{\beta}_0$ and $\hat{\beta}_1$ denote separately the maximum likelihood estimators of β_0 and β_1. Show that $\hat{\beta}_1$ is distributed as $N(\beta_1, \sigma^2/\sum_{i=1}^{n}(x_i - \bar{x})^2)$ and is independent of the sample mean $\bar{Y} = \sum_{i=1}^{n} Y_i$.