1. (9%) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be arbitray vectors in three-dimensional (Euclidean) space. They may be linearly independent or may be linearly dependent.
   (a) Are $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ linearly independent? Why? Does your answer depend upon whether or not $\mathbf{a}$ and $\mathbf{b}$ are linearly independent?
   (b) Are $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$, and $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$ linearly independent? Why? (Prove your answer.) Does your answer depend upon whether or not $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly independent?

2. (12%) Let

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

   and $Q(x_1, x_2, x_3, x_4) = 4x_1^2 - 2x_1x_2 + 4x_2^2 + 4x_3^2 - 2x_3x_4 + 4x_4^2$.
   (a) Find the eigenvalues and normalized eigenvectors of A.
   (b) Discuss whether or not the normalized eigenvectors can be uniquely determined.
   (c) Is $Q(x_1, x_2, x_3, x_4)$ always positive or negative for any real numbers x_1, x_2, x_3, x_4? Why? (Prove your answer.)

3. (12%) Let $f(x) = \cos \pi x$ be a real-valued function defined only on the unit interval, $0 \le x \le 1$.
   (a) Find the Fourier series representation of $f(x)$.
   (b) Find the Fourier sine series representation of $f(x)$.
   (c) Which one of the above two representaions does give a better evaluation for $f(x)$ at $x = 0$? Why?
   (d) Which one of the above two representaions does give a better evaluation for $\frac{df}{dx}(x)$ at $x = 0$? Why?

4. (15%) Solve for $y(x)$ the following initial value problem,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-x},$$
$$y = 1 \text{ at } x = 0,$$
$$\frac{dy}{dx} = 0 \text{ at } x = 0.$$

5. (18%) Solve the following boundary value problem,

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0,$$
$$\frac{\partial f}{\partial r} = 2\cos\theta \text{ for } r = 2,$$
$$f = 3r\cos\theta \text{ as } r \to \infty,$$

   for the function $f(r, \theta)$ defined in the region $2 \le r < \infty, 0 \le \theta < 2\pi$ of a plane, for which (r, θ) is the polar coordinates.

6. (23%) Let z, z_0 be complex variables and $f(z)$ be a complex function.
   (a) (15%) Evaluate the integral $\int_C (z - z_0)^n dz$, (n = integer), along the circle C with center at z_0 and radius r described in the counterclockwise direction.
   (b) (8%) Find $\int_C f(z)dz$ if $f(z) = k$ (a constant), z, $\frac{1}{z}$, $\frac{2\sinh^2 z + 3\cosh 3z}{z}$, respectively, where C is any simple closed contour having $z_0 = 0$ in its interior, and C is taken in the positive direction.

7. (11%) Find the extremals for the following functionals:
   (a) $v(y(x)) = \int_2^3 y^2(1 - \frac{dy}{dx})^2 dx$ with $y(2) = 1$ and $y(3) = 3$;
   (b) $v(y(x), z(x)) = \int_0^1 y'z'dx$ with $y(0) = 0$, $y'(0) = 1$, $z(0) = 0$, and $z'(0) = 1$.