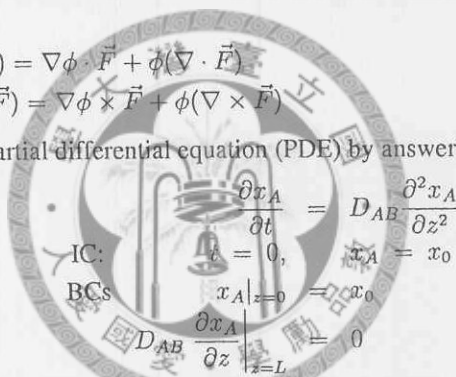


- (10%) Find a second-order differential equation which has $y = ae^x + b \cos x$ as a general solution, where a and b are constants.
- (10%) Find the solution of the equation $\frac{d^2 y(t)}{dt^2} - 4 \frac{dy(t)}{dt} + 4y(t) = 0$ for which $y(t=0) = 3$, $\left. \frac{dy(t)}{dt} \right|_{t=0} = 4$.
- (20%) A tank is initially filled with 100 m^3 of salt solution containing 1 kg of salt per cubic meter. Fresh brine containing 2 kg of salt per cubic meter runs into the tank at the rate of $5 \text{ m}^3/\text{min}$, and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t , and determine how long it will take for this amount to reach 150 kg.
- \vec{F} and $\phi(x, y, z)$ are vector field and scalar field, respectively. If \vec{F} can be expressed as $f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$. Prove that

(a) (10%) $\nabla \cdot (\phi \vec{F}) = \nabla \phi \cdot \vec{F} + \phi (\nabla \cdot \vec{F})$

(b) (10%) $\nabla \times (\phi \vec{F}) = \nabla \phi \times \vec{F} + \phi (\nabla \times \vec{F})$

- Solve the following partial differential equation (PDE) by answering (a), (b), and (c).



$$\frac{\partial x_A}{\partial t} = D_{AB} \frac{\partial^2 x_A}{\partial z^2}$$

IC: $x_A = 0, \quad x_A = x_0$

BCs: $x_A|_{z=0} = x_0, \quad D_{AB} \frac{\partial x_A}{\partial z} \Big|_{z=L} = 0$

- (5%) Please change the above PDE, IC, and BCs using the following variables: $\theta = x_A/x_0$, $\xi = z/L$, $\tau = D_{AB}t/L^2$
 - (5%) Solve the PDE of (a) using the method of separation of variables (Note: please express your answer by $\theta(\xi, \tau)$)
 - (5%) Use the boundary condition to show that the eigenvalues are $\lambda_n = \left(\frac{2n-1}{2}\right)\pi, n = 1, 2, 3, \dots$
 - (5%) Obtain the $x_A(z, t)$ from the orthogonality
- (a) (10%) Show that the partial differential equation

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu \right)$$

can be transformed to a standard heat equation $\left(\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \right)$ by choosing α and β appropriately and letting $u(x, t) = e^{\alpha x + \beta t} v(x, t)$

- (10%) Use the idea of (a) to solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial u}{\partial x} \right) \quad \text{for } 0 < x < 4, \quad t > 0 \\ u(0, t) &= u(4, t) = 0 \quad \text{for } t \geq 0 \\ u(x, 0) &= 1 \end{aligned}$$