1. (10%) Find a second-order differential equation which has $y = ae^x + b\cos x$ as a general solution, where a and b are constants.

2. (10%) Find the solution of the equation $\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} + 4y(t) = 0$ for which $y(t=0) = 3$, $\left.\frac{dy(t)}{dt}\right|_{t=0} = 4$.

3. (20%) A tank is initially filled with 100 m^3 of salt solution containing 1 kg of salt per cubic meter. Fresh brine containing 2 kg of salt per cubic meter runs into the tank at the rate of 5 m^3/min, and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t, and determine how long it will take for this amount to reach 150 kg.

4. $\vec{F}$ and $\phi(x,y,z)$ are vector field and scalar field, respectively. If $\vec{F}$ can be expressed as $f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$. Prove that
   (a) (10%) $\nabla \cdot (\phi\vec{F}) = \nabla\phi \cdot \vec{F} + \phi(\nabla \cdot \vec{F})$
   (b) (10%) $\nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi(\nabla \times \vec{F})$

5. Solve the following partial differential equation (PDE) by answering (a), (b), and (c).

$$\frac{\partial x_A}{\partial t} = D_{AB}\frac{\partial^2 x_A}{\partial z^2}$$
$$\text{IC:} \quad t = 0, \quad x_A = x_0$$
$$\text{BCs} \quad x_A|_{z=0} = x_0$$
$$D_{AB}\left.\frac{\partial x_A}{\partial z}\right|_{z=L} = 0$$

   (a) (5%) Please change the above PDE, IC, and BCs using the following variables: $\theta = x_A/x_0,\ \xi = z/L,\ \tau = D_{AB}t/L^2$
   (b) (5%) Solve the PDE of (a) using the method of separation of variables (Note: please express your answer by $\theta(\xi,\tau)$)
   (c) (5%) Use the boundary condition to show tat the eigenvalues are $\lambda_n = \left(\frac{2n-1}{2}\right)\pi, n = 1,2,3\ldots$
   (d) (5%) Obtain the $x_A(z,t)$ from the orthogonality

6. (a) (10%) Show that the partial differential equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right)$$

   can be transformed to a standard heat equation $\left(\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}\right)$ by choosing α and β appropriately and letting $u(x,t) = e^{\alpha x + \beta t}v(x,t)$
   (b) (10%) Use the idea of (a) to solve

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial u}{\partial x}\right) \quad \text{for } 0 < x < 4,\ t > 0$$
$$u(0,t) = u(4,t) = 0 \quad \text{for } t \ge 0$$
$$u(x,0) = 1$$