1. (30 %). Let M be a positive-definite symmetric $(n \times n)$ real matrix; i.e., $\mathbf{x}^T \mathrm{M}\mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$ where $\mathbf{x}$ is the $n \times 1$ column vector with real entries.

   (a). (10%). Show that all the eigenvalues of M are positive.

   (b). (10%). Consider the following iterative process: with an initially given unit vector $\mathbf{v}_1$, for $i = 1, 2, 3, ...,$

   (i) Compute $\mathbf{u}_i = \mathrm{M}\mathbf{v}_i$,

   (ii) Find $\mathbf{v}_{i+1} = \mathbf{u}_i / \|\mathbf{u}_i\|$,

   where $\|\mathbf{u}_i\|$ denotes the magnitude of $\mathbf{u}_i$. Show that if the above process converges, the sequence $\{\mathbf{v}_i,\ i = 1, 2, 3, ...\}$ converges to an eigenvector of M with the associated eigenvalue approached by $\{\|\mathbf{u}_i\|,\ i = 1, 2, 3, ...\}$. (Note that you may assume $n = 2$ for partial credits.)

   (c). (10%). Solve the following problem

$$\max_{\mathbf{x} \in \mathbf{R}^n} \mathbf{x}^T \mathrm{M}\mathbf{x}, \qquad \text{subject to the constraint} \quad \|\mathbf{x}\| = 2,$$

   in terms of the eigenvector and eigenvalue of M. (Note that you may assume $n = 2$ for partial credits.)

2. (35%).

   (a). Let a be a positive constant and $f(t)$ be a continuous function on $[0,\ L]$, $L > 0$.

   (i). (5%). Show that the only solution of

$$ty' + ay = 0$$

   which is bounded as $t \to 0^+$ is the trivial solution.

   (ii) (10%). Let $f(0) = b$. Show that the equation

$$ty' + ay = f(t)$$

   has a unique solution which is bounded as $t \to 0^+$ and find the limit of this solution as $t \to 0^+$.

   (b). (20%). Given that the equation

$$ty'' - (2t + 1)y' + 2y = 0 \qquad (t > 0)$$

   has a solution of the form e^{ct} for some c, find the **general solution**.

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3. (35%). Consider the Poisson equation

$$\nabla^2\varphi(x,y,z) = f(x,y,z)$$

in a bounded domain Ω enclosed by the surface S with unit outward normal $\hat{\mathbf{n}}$.

(a). (15%). If the boundary condition is of the Neumann type; i.e., $\frac{\partial\varphi}{\partial n} = g(x,y,z)$ on the surface S, show that

$$\int_{\Omega} f(x,y,z)dV = \int_{S} g(x,y,z)ds,$$

where dV and ds are the infinitesimal volume and surface elements, respectively. Also provide the physical interpretation of the above equation.

(b). (20%). For simplicity, consider a two-dimensional case in a unit square domain $\Omega = (0,1) \times (0,1)$ as shown in Figure 1. The source term $f(x,y) = 1,\ \forall\,(x,y) \in \Omega$, and the boundary conditions are of the Dirichlet type: $\varphi = -1$ on $x = 0, x = 1$ and $y = 1$; but $\varphi = 0$ on $y = 0$. Solve for $\varphi(x,y)$ and evaluate the normal derivatives of $\varphi(x,y)$ along the boundaries.

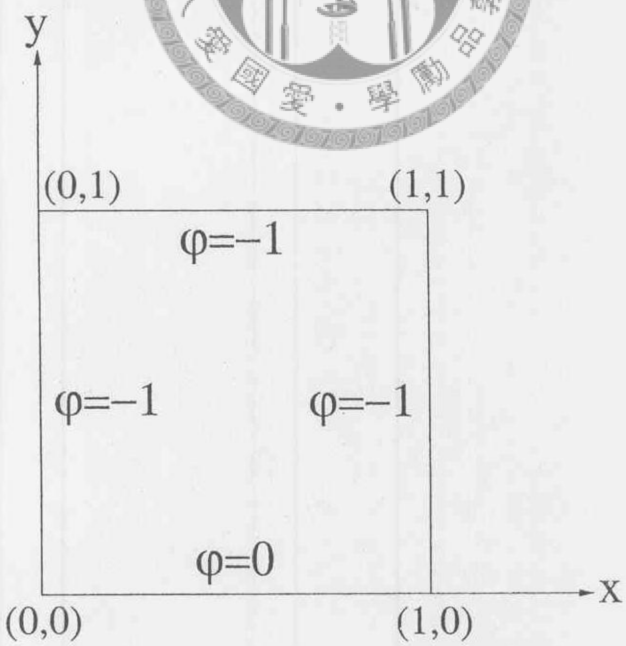

Figure 1: Problem 3(b).

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