1. The equation of motion of a pendulum subjected to forced oscillations is:

$$Mk^2\ddot{\theta} = Mah\omega^2 \cos\theta \sin\omega t - Mgh\sin\theta$$

where θ is the angle of inclination of the pendulum to the downward vertical at time t, and the parameters M, k, a, h, ω, g are constants.

(a) This equation reduces to a linear, second order differential equation for small values of θ. Write down the result of linearization with respect to $\theta = 0$. (5%)

(b) Find $\theta(t)$ by solving the linear second order differential equation obtained in (a) given that the pendulum starts from rest in the downward vertical position. (5%)

2. (a) Can we express the general solution of the following differential equation in a power series? Why? (4%)

(b) Find the general solution. (6%)

$$x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + y = 0.$$

3. Consider the following system of the differential equations, and abide by the directions to find the answer.

$$\begin{cases} (D-2)x - y - 6z = 0 \\ (D-2)y - (2D+1)z = 0 \\ (D-2)x - y + (D-8)z = 0 \end{cases}$$

(a) Put the system into the normal form, X' = AX and find the general solution. (12%)

(b) Suppose $x(0) = y(0) = z(0) = 0$. Please find the Laplace transforms of x, y and z. (8%)

4. The population changes between City A and City B can be described as follows: 15% of those living in City A will move to City B and 3% of those living in City B will move to City A. For simplicity, we assume that the population remains stable, i.e., the sum of the population of City A and City B remains constant. Suppose that there are now 500 thousand people living in City A and 700 thousand people living in City B. Please answer the following questions:

(a) What are the populations of City A and City B, respectively, in the next year? (5%)

(b) What are the populations of City A and City B, respectively, in the k-th year? (7%)

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(c) What are the populations of City A and City B, respectively, when k is infinity? (8%)

5. Solve the following differential equation (10%)

$$\frac{dy}{dx}=\frac{y^2+2y}{y^4+2xy+4x}$$

6. Use LU decomposition to solve the following system of linear equations (10%)

$$-x_1+2x_2-x_3+3x_4=6$$
$$x_1-4x_2+5x_3-5x_4=-7$$
$$-2x_1+6x_2-5x_3+7x_4=7$$
$$-x_1-4x_2+11x_3-2x_4=7$$

7. For the linear system $A\mathbf{x}=\mathbf{b}$ with

$$A=\begin{bmatrix}1&1\\1&0\\1&1\\1&1\end{bmatrix} \text{ and } b=\begin{bmatrix}0\\0\\0\\3\end{bmatrix}$$

find:

(a) the normal equation, (3%)

(b) the least squares solution (or solutions) of the system, (3%)

(c) the projection $\mathbf{b}_*$ of $\mathbf{b}$ onto the span of the columns of A, (3%)

(d) the orthogonal projection matrix for the span of the columns of A. (3%)

8. Let $C[-\pi,\pi]$ be an inner product space with

$$\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)g(t)dt.$$

The space P_1 of *linear polynomials* is a subspace of $C[-\pi,\pi]$. Find the projection of the sine function $\sin(t)$ onto the subspace P_1 following the steps below.

(a) Apply the Gram-Schmidt process to the basis $S=\{1,t\}$ of P_1 in order to obtain an orthonormal basis U of P_1. (4 %)

(b) Use the orthonormal basis U to compute the projection of $\sin(t)$ onto P_1. (4%)

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