國立台灣大學九十二學年度碩士班招生考試試題

科目：工程數學(D)

題號：453

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※注意：請於答案卷上標明題號，並依序作答。

1. Consider a special but fair die, which has only three possible outcomes 1, 3 and 5. The coefficients of the quadratic equation $x^2 + bx + c=0$ are determined by tossing this fair die twice (the first outcome is b, the second one is c).

1.1) Write down the sample space of this die tossing experiment. (4%)

1.2) Express the event that the quadratic equation has real roots in terms of the sample space. (3%)

1.3) Find the probability that the equation has real roots. (3%)

2. Consider $n+m$ trials having a common probability of success p. The value of p is not known in advance but is chosen from a uniform (0,1) population. What is the conditional distribution of p given that n successes have been observed from in the $n+m$ trials, where n and m are positive integers? (10%)

3. Suppose that if a signal value **s** is sent from location A, then the signal value received at location B is normally distributed with parameters (**s**, 1). Let **S**, the value of the signal sent at A, be normally distributed with parameters (μ, σ^2) and **R** be the value received at B.

3.1) Derive the joint p.d.f. f_{SR}. (7%)

3.2) Compute E[**R**], Var[**R**] and Cov(**R**, **S**)(9%)

4. Suppose that a student takes a 20 question true-false exam, for which he is totally unprepared. Thus the student will guess at the answer to every question and the probability the student is right, for each question, is 1/2. What is the probability that the student gets 8 to 12 correct answers?

4.1) Derive the equation, not the number, of the exact probability. (7%)

4.2) Approximate the probability by using Φ( ) function, where

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)dz. \text{ (7\%)}$$

5. (20%) Label the following statements as being true or false. (No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%):

(a) Let $A \in \mathcal{R}^{n\times n}$. If the rows of A are orthogonal, then the columns of A are also orthogonal.

(b) Let $A \in \mathcal{R}^{m\times n}$ and R be its reduced row echelon form. Then there is a unique invertible $m \times m$ matrix P such that $PA = R$.

(c) There does not exist any system of linear equations with exactly 2 solutions.

(d) Similar matrices *always* have the same eigenvalues.

(e) Let **u** and **v** be eigenvectors of a symmetric matrix and $\mathbf{u} \neq c\,\mathbf{v}$ for any scalar c. Then **u** and **v** are orthogonal.

(f) Let $W = \{A \in \mathcal{R}^{n\times n} : A \text{ is not invertible}\}$. Then W is a subspace of $\mathcal{R}^{n\times n}$ under the operations of matrix addition and scalar multiplication defined on $\mathcal{R}^{n\times n}$.

(g) An $n \times n$ matrix is invertible if and only if its reduced row echelon form is I_n.

(h) Let $\mathcal{S}$ be a nonempty finite subset of $\mathcal{R}^n$ and $V = Span\mathcal{S}$. If every vector in V can be *uniquely* expressed as a linear combination of vectors in $\mathcal{S}$, then $\mathcal{S}$ is linearly independent.

(i) Let $\mathcal{S}$ be a nonempty finite subset of $\mathcal{R}^n$ and suppose that $\mathcal{S}$ spans $\mathcal{R}^n$. If $<\mathbf{x}, \mathbf{v}>= 0$ for every vector **v** in $\mathcal{S}$, then $\mathbf{x} = \mathbf{0}$.

(j) Let $\mathcal{S}$ be a nonempty finite subset of $\mathcal{R}^n$ and suppose that $\mathcal{S}$ spans $\mathcal{R}^n$. Let $T\, U : \mathcal{R}^n \to \mathcal{R}^m$ be linear. If $T(\mathbf{v}) = U(\mathbf{v})$ for every $\mathbf{v} \in \mathcal{S}$, then $T = U$.

6. (10%) Let $A = I_n + c\,\mathbf{v}\mathbf{v}^T$ where c is a scalar and $\mathbf{v} \in \mathcal{R}^{n\times 1}$. Suppose that $\mathbf{v}^T\mathbf{v} = 1$, find det A in terms of c. (Hint: Consider the eigenvectors of A).

7. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 4 \end{bmatrix}.$$

(a) (10%) Let $W = \text{Span}\ \mathcal{B}$. Find the vector in W that is closest to **v**.

(b) (10%) Find an orthogonal basis for $W^{\perp}$.