(1) (10%) Let a, b, c be positive integers, (a, b, c) be the great common divisor of a, b, c and $[a, b, c]$ the least common multiple of a, b, c. Show that

$$(a, b, c)[a, b, c] = abc$$

(2) (10%) Find all integers that satisfy the following congruences simultaneously: $x \equiv 1 (\mathrm{mod}\ 2)$, $x \equiv 2 (\mathrm{mod}\ 3)$, $x \equiv 3 (\mathrm{mod}\ 5)$.

(3) (10%) Let V be the $\mathbb{R}$-vector space of all real valued functions defined on $\mathbb{R}$ If for each real number α the function g_α is defined by $g_\alpha(x) = \cos(x + \alpha)$ for all $x \in \mathbb{R}$, what is the dimension of the subspace of V generated by $\{g_\alpha : \alpha \in \mathbb{R}\}$?

(4) (20%) Let A and B be square matrices over a field F. We say that A is similar to B if there exists an invertible matrix P over F such that $B = P^{-1}AP$. We say that A is equivalent to B if there exist an invertible matrix P over F and an invertible matrix Q over F such that $B = Q^{-1}AP$.

(a) Show that if A and B are similar, then A^m and B^m are similar for all positive integers m.

(b) If A and B are equivalent, is A^m equivalent to B^m ? Prove your answer.

(5) (20%) Let G be a group and H, K be subgroups of G.

(a) Give an example to show that HK need not be a subgroup of G.

(b) Let K be a normal subgroup of G. Show that HK is a subgroup of G.

(6) (20%) Let R be a ring with identity and I, J be two-sided ideals of R.

(a) Give an example to show that $IJ = I \cap J$ need not to be true.

(b) Suppose that $I + J = R$. Show that $IJ = I \cap J$.

(7) (10%) Let F be a field whose characteristic is not 2, and let a and b be elements of F such that $x^2 - a$ and $x^2 - b$ are irreducible over F. If K is a splitting field of $f = (x^2 - a)(x^2 - b)$ over F, show that $[K : F]$ is 2 or 4 according as $x^2 - ab$ is reducible or irreducible over F.