1. This problem is about Newton's method for solving nonlinear equation $f(x) = 0$, where f is a smooth function. Answer the following questions.
   (a) (5%) What is Newton's method?
   (b) (5%) Write a pseudo-code for Newton's method (no longer than 20 lines).
   (c) (5%) Show that Newton's method converges quadratically if the starting value is closed to the root.
   (d) (5%) Give an example to demonstrate that Newton's method may diverge.

2. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define vector norms $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ and $\|x\|_\infty = \max_i |x_i|$. For a matrix $A = (a_{ij})_{n\times n}$, we define the operator norm
$$\|A\|_p := \sup_{\|x\|_p=1} \|Ax\|_p.$$
   (a) (5 %) Show that $\|A\|_1 = \max_j \sum_i |a_{ij}|$.
   (b) (5 %) Show that $\|A\|_\infty = \max_i \sum_j |a_{ij}|$.
   (c) (5 %) Show that $\|A\|_2 = \max_i |\lambda_i(A)|$, where $\lambda_i(A)$ are the eigenvalues of A.
   (d) (5 %) Define the condition number of A by $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$. Find the condition numbers $\kappa_p(A)$, $p = 1, 2, \infty$ for the matrix
$$A = \begin{pmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{pmatrix}$$
   where ϵ is a small number.

3. (10%) Derive and prove the following Simpson rule for numerical integration:
$$\int_{-h}^{h} f(x)\,dx = \frac{h}{3}(f(-h) + 4f(0) + f(h)) + O(h^5).$$

4. (10%) Let h be a small mesh size. Let us abbreviate $f(ih)$ by f_i. Suppose we are given f_i, $i = -1, 0, 1, 2$. Ues them to find best approximations for $f'(\alpha h)$ and $f''(\alpha h)$, where $0 < \alpha < 1$. Find the corresponding approximation errors and prove your statement.

5. We call a matrix $A = (a_{ij})_{n\times n}$ diagonally dominant if for each $1 \le i \le n$, $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$.
   (a) (5%) What is the Jacobi method for solving the linear equation $Ax = b$?
   (b) (5%) Write a pseudo-code for the Jacobi method.
   (c) (10%) If A is diagonally dominant, show that the Jacobi method converges.
   (d) (10%) Write anything you know about the Jacobi method, for instance, the error reduction rate, the Jacobi method for Poisson equation, etc.

6. (10%) Write a second-order Runge-Kutta method for solving the ODE $\dot{x} = f(t, x)$. Prove your statement.