1. Let T be a random variable with a Student's t distribution with p degrees of freedom.

(1a) (10%) Show that $T \xrightarrow{d} Z$, as $p \to \infty$, where Z is a standard normal random variable.

(1b) (10%) Show that $U = T^2/(p + T^2)$ has a beta distribution with parameters $1/2$ and $p/2$.

2. Let X_1 and X_2 be independent random variables from $N(0,1)$, and $W = X_1/X_2$.

(2a) (5%)(5%) Show that the expectation of W does not exist, and find the median of W.

(2b) (10%) Let $Y_1 = a_1X_1 + a_2X_2 + a_3$ and $Y_2 = b_1X_1 + b_2X_2 + b_3$. Find a set of coefficients $(a_1, a_2, a_3, b_1, b_2, b_3)$ such that (Y_1, Y_2) follows a bivariate normal with $E[Y_1] = 1$, $E[Y_2] = 2$, $Var[Y_1] = 1$, $Var[Y_1] = 4$, and $Cov(Y_1, Y_2) = 1$.

3. Let $X_1, \cdots, X_n$ be a random sample from a population with the probability density function $f(x|\theta) = \exp(-(x-\theta))1_{(x>\theta)}$.

(3a) (8%) Show that $X_{(1)} = \min\{X_1, \cdots, X_n\}$ is a complete sufficient statistic.

(3b) (7%) Show that $X_{(1)}$ is independent of the sample variance $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ with $\bar{X} = \sum_{i=1}^n X_i/n$.

4. Let $X_1, \cdots, X_n$ be a random sample from a population with probability density function $f(x|\lambda) = (1/\lambda)\exp(-x/\lambda)1_{(0,\infty)}(x)$. Moreover, $X_1, \cdots, X_m$ are observed but all we know about $X_{m+1}, \cdots, X_n$ is that they exceed τ.

(4a) (7%) Write down the likelihood function for λ.

(4b) (8%) Show that the maximum likelihood estimator of λ is $\hat{\lambda} = (\sum_{i=1}^m X_i + (n-m)\tau)/m$.

5. (10%) Conditioning on $P = p$, X_1 and X_2 are assumed to be independent and identically distributed $Bernoulli(p)$ random variables. Moreover, let P has a probability density function $f_P(p|\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} p^{\alpha-1}(1-p)^{\beta-1}1_{(0,1)}(p)$, where $\alpha > 0$ and $\beta > 0$. Find α and β such that $E[X_1] = 1/4$ and $Corr(X_1, X_2) = 1/2$.

6. (10%) (10%) Let $X_1, \cdots, X_n$ be a random sample from a population with probability density function

$$f_X(x|\theta) = \frac{1}{(2\pi)^{1/2}\theta x} \exp(\frac{-1}{2}(\frac{\ln(x)}{\theta})^2)1_{(x>0)},$$

where $\theta > 0$, let $T = \sum_{i=1}^n (\ln(X_i))^2$.

(6a) (10%) Show that $P(T > t|\theta_2) \geq P(T > t|\theta_1)$ for all $\theta_2 > \theta_1$ and any constant value t.

(6b) (10%) Show that there is a uniformly most powerful (UMP) test of the null hypothesis $H_0 : \theta \leq \theta_0$ versus the alternative hypothesis $H_1 : \theta > \theta_0$, and find the rejection region of such test.

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