1. (30%) Determine if the following statements are true or false (1 % each) and provide a short proof if it is true or any explanation/counterexample if it is false (2 % each).
   (a) If the only solution to $\mathbf{Ax}=\mathbf{0}$ is $\mathbf{x}=\mathbf{0}$, then the rows of $\mathbf{A}$ are linearly independently.
   (b) If $\mathbf{A}$ and $\mathbf{B}$ are matrices such that $\mathbf{AB}=\mathbf{I}_n$ for some n, then both $\mathbf{A}$ and $\mathbf{B}$ are invertible.
   (c) If $T:R^3 \to R^2$ is linear, then its standard matrix has size 3×2.
   (d) For any square matrix $\mathbf{A}$, $\det \mathbf{A}^T = -\det \mathbf{A}$.
   (e) The dimension of the null space of a matrix equals the rank of the matrix.
   (f) If T is a linear operator on R^n, B is a basis for R^n, $\mathbf{C}$ is the matrix whose columns are the vectors in B, and $\mathbf{A}$ is the standard matrix of T, then $[T]_B = \mathbf{CAC}^{-1}$.
   (g) If an $n\times n$ matrix has n distinct eigenvectors, then it is diagonalizable.
   (h) If $\mathbf{P}$ is an $n\times n$ matrix such that $\det \mathbf{P}=\pm 1$, then $\mathbf{P}$ is an orthogonal matrix.
   (i) In any vector space, $a\mathbf{v}=\mathbf{0}$, where $a\in R, \mathbf{v}\in R^n$, implies that $\mathbf{v}=\mathbf{0}$.
   (j) A matrix representation of a linear operator on $M_{m\times n}$, the set of all $m\times n$ matrices, is an $m\times n$ matrix.

2. (20%) Given a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ as follows: $\mathbf{A}=\begin{bmatrix} 1 & -1 & -2 & -8 \\ -2 & 1 & 2 & 9 \\ 3 & 0 & 2 & 1 \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} -3 \\ 5 \\ -8 \end{bmatrix}$.
   (a) Please find an LU decomposition of $\mathbf{A}$; (10%)
   (b) Please solve $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{x}$ is a 4×1 vector. (10%)

3. (30%) Let T and U be the linear operators on R^3 defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} -x_1+2x_3 \\ x_1+x_2 \\ -x_2+x_3 \end{bmatrix}, \quad \text{and} \quad U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} -2x_1+5x_2-x_3 \\ -3x_1+6x_2-x_3 \\ -x_1+x_2+2x_3 \end{bmatrix},$$

   and let $B=\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$, where $\mathbf{b}_1=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_3=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.
   (a) Find the standard matrices of T and U. (5 %)
   (b) Find $[T]_B$, $[U]_B$ and $[UT]_B$, i.e., the matrix representations of T, U and UT with respect to B, respectively. (15%)
   (c) Determine a relationship among $[T]_B$, $[U]_B$ and $[UT]_B$. (10%)

4. (20%) Given a matrix $\mathbf{A}$ and a set of matrices S as follows:

$$\mathbf{A}=\begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}, \quad S=\left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$$

   (a) Determine if S is a linearly independent subset of $M_{2\times 2}$, the vector space of all 2×2 matrices; (10%)
   (b) Represent the matrix $\mathbf{A}$ as a linear combination of the vectors in the set S. What are the corresponding coefficients? (10%)