Credits will not given if no calculation procedure is included.

1. Consider the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

where

$$P(x) = \frac{p_0}{x} + p_1 + p_2 x + \cdots$$

$$Q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + \cdots$$

p_i and q_i are all real for $i = 0, 1, 2, \ldots$

According to Forbenius Theoirem, there exists at least one solution of the form

$$y = \sum_{n=0}^{\infty} x^{n+r}$$

where r is a real constant.

(a) Find the indicial equation of r. (5%)

(b) If there exist at least one analytic solution, please find the two roots of the indicial equation. (5%)

(c) According to the characteristics of the two roots of the indicial equation, we can classify the solutions. Under what conditions of the two roots, is only one solution analytic? Under what conditions, on the other hand, do there exist possibly two analytic solutions with at least one being analytic? (7%)

(d) Find the exact number of the analytic solutions for the following equations. You do not have to give the solutions but explain why. (8%)

(1) $3xy'' + y' - y = 0$

(2) $4x^2y'' + 4xy' + (4x^2 - 1)y = 0$

(3) $xy'' + y' - 4y = 0$

(4) $y'' + (\cos x)y = 0$

2. For the initial-value problem

$$xy'' - 2y' + xy = 0 \quad \text{subject to } y(0) = 0,$$

please answer the following questions.

(a) Use the Laplace Transform to find the solutions. No credit will be given with other method. (10%)

(b) What does the initial condition $y'(0)$ have to be? And how many solutions are there? Explain your reason. (5%)

3. Find the general solution of the equation (10%)

$$3y'' - 6y' + 6y = e^x \sec(x)$$

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4. A biologist wants to estimate l, the life expectancy of a certain type of insects. He takes a sample of size n and measures the lifetime of each insect. He then finds the average of these numbers. If he believes the lifetimes of these insects are independent random variables with variance of 1.5 days, how large a sample size n should he choose to be 98% sure that his average is accurate within ±0.2 days? Calculate your answers by using the Chebyshev's inequality. (10%)

5. Independent trials, consisting of the flipping of a coin having probability p of coming up heads, are continually performed until either a head occurs or a total of n flips is made. If X denotes the number of times the coin is flipped, calculate the following:
   (a) $P\{X=n\}$ (10%)
   (b) $P(\bigcup_{i=1}^{n}\{X=i\})$ (10%)

6. If X and Y are independent Poisson random variables with respective parameters of λ_1 and λ_2, calculate the following distribution functions
   (a) distribution function of $P\{X+Y=n\}$ (10%)
   (b) conditional distribution function of X,
   given that $X+Y=n$, i.e., $P\{X=k \mid X+Y=n\}$ (10%)