1. Find the polynomial $f = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ with rational coefficients such that $f \equiv 3x+1 \pmod{x(x-1)}$, $f \equiv 9x+1 \pmod{x(x-2)}$ and $f \equiv 27x-23 \pmod{(x-1)(x-3)}$. (20%)

2. Let $G = (\mathbb{R}, +)$, the additive group of real numbers and let H be the subgroup of G formed by all integers. Show that in the quotient group G/H, every finite subgroup is cyclic. (15%)

3. Let $\mathbb{R}$ be the field of real numbers and let $M(2, \mathbb{R})$ be the ring of all 2×2 matrices over $\mathbb{R}$. Denote by S the set of all two-sided ideals of $M(2, \mathbb{R})$. Show that S is formed by exactly two elements. (20%)

4. Let G be a group and let C be a normal subgroup of G. Suppose both C and the quotient group G/C are abelian and C is contained in the center of G. Is it necessary that G is abelian? Prove your answer. (10%)

5. Let $\mathbb{F}$ be a field consisting of two elements. Denote by S the set of all degree 2005 irreducible polynomials over $\mathbb{F}$. Show that the cardinality $|S|$ of S satisfies
$$2005\,|S| = 2^{2005} - 2^{401} - 2^5 + 2$$ (20%)

6. Let $G = (\mathbb{Z}, +)$ be the additive group of integers. Suppose A is a subgroup of $G^n = \underbrace{G \times G \times \cdots \times G}_{n\text{-time}}$, the direct product of n-copies of G, such that for every element x in the quotient group G^n/A and every positive interger N, there is at leat one $y \in G^n/A$ such that $N\cdot y = \underbrace{y + \cdots + y}_{N\text{-time}} = x$. Show that $A = G^n$. (15%)

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