

1. Consider the logistic equation

$$\dot{y} = ry \left(1 - \frac{y}{K}\right), \quad y(0) = y_0$$

where r, K, y_0 are positive constants.

- (a) (10 points) Find its general solutions.
 (b) (10 points) Show that K is an asymptotically stable equilibrium, whereas 0 is an unstable equilibrium.

2. Consider the harmonic oscillator with periodic forcing

$$\ddot{x} + \omega_0^2 x = F \cos \omega t$$

where ω, ω_0, F are real constants.

- (a) (10 points) Find general real solutions for this system.
 (b) (10 points) When will the solution $|x(t)| \rightarrow \infty$ as $t \rightarrow \infty$?

3. If y_1 and y_2 are solutions of $y'' + a(x)y' + b(x)y = 0$. Define $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

- (a) (10 points) Show that $W'(x) = -a(x)W(x)$.
 (b) (5 points) Show that if $W(0) \neq 0$, then $W(x) \neq 0$ for all $x \in \mathbb{R}$.

4. (15 points) Find general solutions for the differential equation

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 \\ \dot{x}_2 &= x_1 + 3x_2 \end{aligned}$$

5. Consider the equation

$$\ddot{x} = x - \frac{3}{2}x^2$$

- (a) (10 points) Show that the quantity $H(x, \dot{x}) := \frac{1}{2}(\dot{x}^2 - x^2 + x^3)$ is unchanged along any trajectories.
 (b) (10 points) Plot the phase diagram of this system. Find its equilibria. Determine their stability types.
 (c) (10 points) Find the separatrix curve on the phase plane.