1. (30 points) Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function having a fixed point x^*.

   (a) What condition determines whether the iteration scheme $x_{k+1} = g(x_k)$ is locally convergent to x^* ?

   (b) What is the convergence rate of the scheme in (a) ? Furthermore, what additional condition implies that the convergence rate is quadratic ?

   (c) Is Newton's method for finding a zero of a smooth function $f : \mathbb{R} \to \mathbb{R}$ an example of such a fixed-point iteration ? If so, what is the function g in this case ? If not, then explain why not.

   (d) Kepler's equation in astronomy reads: $x = y - \varepsilon \sin y$, with $0 < \varepsilon < 1$. Show that for each $x \in [0, \pi]$ there is a y satisfying the equation. (Hint: Interpret this as a fixed point problem).

2. (20 points) Given a set of n data points, $(t_i, y_i), i = 1, 2, \cdots, n$, determining the coefficients x_i of the interpolating polynomial requires the solution of an $n \times n$ system of linear equations $Ax = y$.

   (a) If we use the monomial basis $1, t, t^2, \cdots$, give an expression for the entries a_{ij} of the matrix A that is efficient to evaluate.

   (b) Does the condition of A tend to get better, or worse, or stay about the same as n grows?

   (c) How does this change affect the accuracy with which the interpolating polynomial approximates the given data points ?

3. (15 points) Suppose that $\phi(f, h)$ is a numerical differentiation rule for the derivative of $f'(x)$ and the error series is

$$E(h) = f'(x) - \phi(f, h) = c_1 h + c_3 h^3 + c_5 h^5 + \cdots .$$

   Combine $\phi(f, h)$ and $\phi(f, h/2)$ to find a more accurate approximation to $f'(x)$.

4. (15 points) Derive a quadrature rule of the form

$$\int_1^3 f(x)\, dx = Af(1) + Bf(2) + Cf(3),$$

   with the property that $E(1) = E(x) = E(x^2) = 0$, where $E(f) = \int_1^3 f(x)\, dx - [Af(1) + Bf(2) + Cf(3)]$. What is the order of accuracy of the formula ?

5. (20 points) Suppose that the symmetric matrix

$$B = \begin{bmatrix} \alpha & a^T \\ a & A \end{bmatrix}$$

   of order $n + 1$ is positive definite.

   (a) Show that the scalar α must be positive and the $n \times n$ matrix must be positive definite.

   (b) What is the Cholesky factorization of B in terms of α, a, and A ?