(1) (10%) In the real vector space of all continuous real valued functions, the set $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin^2 x, \cos^2 x\}$ generates a subspace. Find its dimension and prove your answer.

(2) (10%) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection on the xy -plane along the line $L = \{(t, 2t, 3t) | t \in \mathbb{R}\}$.
(a) Find a formula for $T(a, b, c)$.
(b) Find the bases for the kernel and the image of T.

(3) (20%)
(a) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{bmatrix}.$$

Find $\det A$.
(b) Let $f_i(x)$ be a polynomial of degree $i-1$ with leading coefficient a_i and

$$B = \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) & f_4(x_1) & f_5(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & f_4(x_2) & f_5(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) & f_4(x_3) & f_5(x_3) \\ f_1(x_4) & f_2(x_4) & f_3(x_4) & f_4(x_4) & f_5(x_4) \\ f_1(x_5) & f_2(x_5) & f_3(x_5) & f_4(x_5) & f_5(x_5) \end{bmatrix}.$$

Find $\det B$.

(4) (20%)
(a) Let $A, B \in M_{n\times m}(F)$, the set of all $n \times m$ matrices over a field F. Show that

$$\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B).$$

(b) Let $A \in M_{n\times m}(F)$ and $B \in M_{m\times k}(F)$. Show that

$$\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}.$$

(5) (40%) Let $A \in M_n(\mathbb{R})$ be a skew-symmetric matrix.
(a) Find $\langle A\mathrm{v}, \mathrm{v}\rangle$, where $\mathrm{v} \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.
(b) Show that every nonzero eigenvalue of A is pure imaginary.
(c) Show that $I_n + A$ is invertible, where I_n is the identity matrix.
(d) Show that $B := (I_n - A)(I_n + A)^{-1}$ is an orthogonal matrix.
(e) Show that B does not have -1 as an eigenvalue.