1. (20%) Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ 0 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} 89 & a_{12} & 0 \\ a_{21} & 61 & 0 \\ a_{31} & 0 & 1 \end{bmatrix}.$$

   (a) For the real vectors $\mathbf{x}$ and $\mathbf{y}$, are there some conditions which should be imposed upon x_1, x_2, x_3, y_1 such that $\mathbf{x} \times \mathbf{y} + \mathbf{y} \times \mathbf{x} = \mathbf{0}$? What conditions if there are? Why if there are not?

   (b) For the real matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution? What conditions if there are? Why if there are not?

   (c) For the real matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that A, A^2, A^3 are linearly dependent? What conditions if there are? Why if there are not?

   (d) For the complex matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that A has real eigenvalues? What conditions if there are? Why if there are not?

2. (13%) State Green's theorem in the plane and show that the fundamental theorem of calculus $\int_a^b \frac{dF}{dx} dx = F(b) - F(a)$ for a function $F(x)$ defined over $a \leq x \leq b$ can be deemed as a special case of the Green theorem for functions $P(x, y)$ and $Q(x, y)$ defined over $a \leq x \leq b$ in the plane. Are there some conditions which should be imposed upon $F(x), P(x, y), Q(x, y)$ such that the theorems are valid? What conditions if there are? Why if there are not?

3. Find *all* solutions for the following differential equations:

   (a) (6%) $\frac{du}{dx} = \frac{u}{2x}$;

   (b) (5%) $\frac{d^2u}{dx^2} = \frac{1}{2x}\frac{du}{dx}$;

   (c) (11%) $x\frac{\partial u}{\partial x} = 2y\frac{\partial u}{\partial y}$;

   (d) (11%) $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial y^2}$.

4. (24%) Complex analysis.

   (a) Locate all singularities for the following functions: (i) $f_1(z) = \text{Arg } z$, (ii) $f_2(z) = |z|$, (iii) $f_3(z) = \frac{z^3 - 2z^2}{3z^2 + 4z - i}$, (iv) $f_4(z) = \text{Re } z$, and determine which are isolated.

   (b) Find the branch points and branch cuts for the following functions: (i) $g_1(z) = \text{Log }(3 - 2z)$, (ii) $g_2(z) = \text{Log }(3z - 2 + 4i)$, (iii) $g_3(z) = \text{Log}_{-\frac{\pi}{2}} (4 - 2z)$, (iv) $g_4(z) = \text{Log}_2 (2z + 1)$, and plot the required answers on the z planes.

5. (10%) Find the extremal for the functional $v(y(x)) = \int y^2 (1 - \frac{dy}{dx})^2 dx$ with $y(2) = 1$ and $y(3) = 3$, and plot the required extremal on the xy plane.