※注意：請於答案卷上依序作答，並應註明作答之題號。

Note that for problems 1-15, we consider only matrices with real values.

1. (3%) What is the determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}?$$

   (A) 70 (B) -70 (C) 65 (D) -65 (E) 85

2. (3%) If $\mathbf{v} = [1, 2, 3, 4, 5]^T$, what is the rank of $\mathbf{v}\mathbf{v}^T$?

   (A) 1 (B) 2 (C) 3 (D) 0 (E) Not well defined

3. (3%) For the same $\mathbf{v}$ in problem 2, what is the rank of $\mathbf{v}^T\mathbf{v}$?

   (A) 1 (B) 2 (C) 3 (D) 0 (E) Not well defined

4. (3%) Which of the following is incorrect?

   (A) Any $n+1$ different nonzero points in R^n must be linearly dependent.

   (B) In R^2 any two linearly dependent vectors must be on a straight line.

   (C) Let A be an $m \times n$ matrix with $m > n$ and $\mathbf{b}$ be an $m \times 1$ vector. Then $A\mathbf{x} = \mathbf{b}$ may have multiple solutions.

   (D) Let A be an $m \times n$ matrix with $m < n$ and $\mathbf{b}$ be an $m \times 1$ vector. Then $A\mathbf{x} = \mathbf{b}$ must have at least one solution.

   (E) Let A be any 5×4 matrix and I be the 5×5 identity matrix. Then $I + AA^T$ is invertible.

5. (3%) Which of the following is incorrect?

   (A) If A, B are invertible, then $(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$

   (B) $\text{trace}((A+B)C) = \text{trace}(AC) + \text{trace}(BC)$

   (C) $\det(AB) = \det(A)\det(B)$

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(D) trace(AB) = trace(A)trace(B)

(E) If A, B are positive definite, then so is $A + B$

6. (3%) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Which of the following symmetric matrix B satisfies

$$\mathrm{x}^T A \mathrm{x} = \mathrm{x}^T B \mathrm{x}, \forall \mathrm{x} \in R^3?$$

(A) $\begin{bmatrix} 1 & 4 & 7 \\ 4 & 5 & 6 \\ 7 & 6 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 4 & 6 \\ 4 & 5 & 6 \\ 6 & 6 & 9 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (E) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$

7. (3%) Given an $n \times n$ matrix A written as

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where B is $n_1 \times n_1$, C is $n_1 \times n_2$, D is $n_2 \times n_2$, 0 is $n_2 \times n_1$, and $n = n_1 + n_2$. Assume B and D are invertible and 0 is a zero matrix. Then what is A^{-1}?

(A) $\begin{bmatrix} B^{-1} & C^{-1} \\ 0 & D^{-1} \end{bmatrix}$ (B) $\begin{bmatrix} B^{-1} & 0 \\ C^{-1} & D^{-1} \end{bmatrix}$ (C) $\begin{bmatrix} B^{-1} & 0 \\ -B^{-1}CD^{-1} & D^{-1} \end{bmatrix}$

(D) $\begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$ (E) A may not be always invertible.

8. (3%) Given

$$A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}.$$

Let $\lambda_1 > \lambda_2$ be A's eigenvalues. What is $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$?

(A) $\begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$ (B) $\begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix}$ (C) $\begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$ (D) $\begin{bmatrix} 2 \\ 0.3 \end{bmatrix}$ (E) $\begin{bmatrix} 3 \\ 0.5 \end{bmatrix}$

9. (3%) Following problem 8, what are A's eigenvectors associated with $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$? Each eigenvector is scaled to have length one.

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(A) $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$ (B) $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$ (C) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

(E) $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

10. (3%) Assume the answer of problem 9 are $\mathbf{v}_1, \mathbf{v}_2$. If $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and we represent it as

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2,$$

then what are α_1 and α_2?

(A) $(\sqrt{3}, -\sqrt{3})$ (B) $(2\sqrt{2}, -2\sqrt{2})$ (C) $(2\sqrt{2}, 2\sqrt{2})$ (D) $(\sqrt{3}, \sqrt{3})$ (E) $(\sqrt{3}, 2\sqrt{3})$

11. (4%) Following problems 8-10, what is

$$\lim_{n\to\infty} A^n \mathbf{x}?$$

(A) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (B) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (C) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$ (E) $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

12. (4%) Consider an $n \times n$ symmetric matrix A. If A is positive definite, then which of the following four properties is wrong? If you think all are correct, select E.

(A) $A_{ii} > 0, \forall i$

(B) A is invertible

(C) $A_{ii} + A_{jj} - 2A_{ij} > 0, \forall i \neq j$

(D) $A_{ii}A_{jj} - A_{ij}^2 > 0, \forall i \neq j$

(E) All are correct.

13. (4%) If

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

what is the solution of

$$A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1.5\\7.5\\15.5\\14.0\\4.0\end{bmatrix}?$$

(A) $\begin{bmatrix}0.5\\0.5\\1.5\\2.5\\-0.5\end{bmatrix}$ (B) $\begin{bmatrix}-0.5\\-1.5\\0.5\\1.5\\2.5\end{bmatrix}$ (C) $\begin{bmatrix}1.5\\-0.5\\-1.5\\0.5\\1.5\end{bmatrix}$ (D) $\begin{bmatrix}1.5\\2.5\\-0.5\\-1.5\\0.5\end{bmatrix}$ (E) $\begin{bmatrix}0.5\\1.5\\2.5\\-0.5\\-1.5\end{bmatrix}$

14. (4%) Given any vectors $\mathbf{x}_1,\ldots,\mathbf{x}_l\in R^n$. Define an $l\times l$ matrix A with $A_{ij}\equiv\mathbf{x}_i^T\mathbf{x}_j, i,j=1,\ldots,l$. Which of the following is incorrect?

(A) A is symmetric

(B) A is positive semi-definite (i.e., A's eigenvalues are ≥ 0)

(C) A is invertible

(D) A's diagonal elements are nonnegative

(E) A is a square matrix

15. (4%) Define

$$f(\mathbf{x})\equiv\frac{1}{2}\begin{bmatrix}x_1 & \ldots & x_5\end{bmatrix}\begin{bmatrix}1&2&3&4&5\\2&3&4&5&6\\3&4&5&6&7\\4&5&6&7&8\\5&6&7&8&9\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_5\end{bmatrix}$$

and

$$\nabla f(\mathbf{x})\equiv\begin{bmatrix}\frac{\partial f(x_1,\ldots,x_5)}{\partial x_1}\\\vdots\\\frac{\partial f(x_1,\ldots,x_5)}{\partial x_5}\end{bmatrix}.$$

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What is

$$\nabla f\left(\begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}\right)?$$

(A) $\begin{bmatrix}22\\25\\33\\39\\45\end{bmatrix}$ (B) $\begin{bmatrix}22\\28\\36\\40\\51\end{bmatrix}$ (C) $\begin{bmatrix}22\\25\\33\\40\\46\end{bmatrix}$ (D) $\begin{bmatrix}22\\28\\34\\40\\46\end{bmatrix}$ (E) $\begin{bmatrix}22\\24\\35\\39\\45\end{bmatrix}$

16. (5%) The maximum height of a binary tree with n nodes is _________.

17. (5%) A nonempty subset I of a ring $(R, +, \cdot)$ is called an ideal of R if for all $a, b \in I$ and all $r \in R$: $a - b \in I$, $a \cdot r \in I$, and $r \cdot a \in I$. If $(R, +, \cdot)$ is furthermore a field, then it has _________ ideals.

18. (10%) If $a \in Z_n$ has a multiplicative inverse (i.e., there exists a b such that $a \cdot b = b \cdot a = 1$), then a is called a unit. For any positive integer $n > 1$, there are _________ units in Z_n.

19. (5%) Let K_n^* be a *directed* graph with n nodes. If for each distinct pair x, y of nodes, either $(x, y) \in K_n^*$ or $(y, x) \in K_n^*$ but not both, then K_n^* is called a tournament. A directed graph (V, E) is transitive if $(a, b) \in E \wedge (b, c) \in E$ imply $(a, c) \in E$. There are _________ transitive tournaments on n players.

20. (10%) Consider a binary string $x_1x_2\cdots x_n$. The weight of $x_1x_2\cdots x_n$ is defined as $\sum_i x_i$. There are 2^n strings. Among them, _________ have even weight.

21. (5%) Consider $x_1 + x_2 + \cdots + x_n < r$, where $x_i \geq 0$ for $1 \leq i \leq n$. the number of nonnegative integer solutions is _________.

22. (5%) The generating function for $1, 2, 3, 4, \ldots$ is _________.

23. (5%) There are _________ functions from $\{0,1\}^m$ to $\{0,1\}^n$.