\# 1. (20 pts) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of non-negative real numbers satisfying

$$x_{n+1} \le x_n + \frac{1}{n^2}, \quad y_{n+1} \le y_n + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Can the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ always exist, respectively? Prove or disprove your answer.

\# 2. (30 pts) Let f be a continuous real-valued function on $\mathbb{R}$ such that the improper Riemann integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. Define the function g on $\mathbb{R}$ by

$$g(y) = \int_{-\infty}^{\infty} f(x) \cos(xy)\, dx.$$

(i) Must the function g be continuous? (20 pts)

(ii) Can the limit $\lim_{y\to\infty} g(y)$ always exist? (10 pts)

Prove or disprove all your answers.

\# 3. (10 pts) Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy "$f(K)$ is compact whenever K is compact subset of $\mathbb{R}^n$". Must f be continuous? Prove or disprove your answer.

\# 4. (20 pts) Let u and v be two real-valued C^1 functions on $\mathbb{R}^2$ such that the gradient ∇u is never zero, and such that, at each point, ∇u and ∇v are linearly dependent vectors. Given $p = (x_0, y_0) \in \mathbb{R}^2$. Must there exist a C^1 function F of one variable such that $v(x,y) = F(u(x,y))$? Prove or disprove your answer.

\# 5. (20 pts) Given $h : \mathbb{R} \to \mathbb{R}$ a nonzero smooth function with compact support i.e. the closure of $\{x \in \mathbb{R} : h(x) \neq 0\}$ is compact. For $\epsilon > 0$, let

$$u_\epsilon(x) = \frac{\int_{-\infty}^{\infty} (x-y)\, e^{-K(x,y)/\epsilon}\, dy}{\int_{-\infty}^{\infty} e^{-K(x,y)/\epsilon}\, dy}, \quad \forall x \in \mathbb{R},$$

where $K(x,y) = \frac{1}{4}(x-y)^2 + \frac{1}{2}h(y)$ for $x, y \in \mathbb{R}$. Answer the following questions:

(1) Can each u_ϵ be a smooth function with compact support? (10 pts)

(2) Can the limit $\lim_{\epsilon\to 0+} u_\epsilon(x)$ always exist? in what sense? (10 pts)

Prove or disprove all your answers.