國立臺灣大學95學年度碩士班招生考試試題

1. (20 points) Give a brief explaination of the following terms:
   (a) Machine epsilon (or called machine precision, or unit roundoff)
   (b) Inverse interpolation for finding a root of a nonlinear equation $f(x) = 0$
   (c) Condition number of a square nonsingular matrix A
   (d) Vandermonde matrix

2. (15 points) Write out the LU factorization of the following matrix (show both the L and U matrices explicitly)

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

3. (15 points) Given a set of 3 data points: $(t_1, y_1) = (-2, 27)$, $(t_2, y_2) = (0, -1)$, $(t_3, y_3) = (1, 0)$.
   (a) Write down the Lagrange form for the polynomial of degree two, $p_2(t)$, interpolating these three points.
   (b) As in (a) above, write down the Newton form of the interpolating polynomial.

4. (10 points) In approximating the first derivative of a function $f : \mathbb{R} \to \mathbb{R}$, the forward difference formula

$$f'(x) \approx D_+^h(x) = \frac{f(x+h) - f(x)}{h}$$

and the backward difference formula

$$f'(x) \approx D_-^h(x) = \frac{f(x) - f(x-h)}{h}$$

are both first-order accurate, meaning that their dominant error terms are $O(h)$. Show how these two formulas can be combined to produce a difference approximation for the first derivative of f that is second order accurate.

5. (20 points) Devise a (Gaussian) quadrature rule of the form

$$\int_{-1}^{1} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2),$$

where the nodes x_i and weights w_i, $i = 1, 2$, are chosen to maximize the polynomial degree of the resulting quadrature rule.

6. (20 points) The centered difference approximation

$$y' \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

leads to the two-step leap-frog method

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n), \quad n = 2, 3, \cdots$$

for solving the ordinary differential equation $y' = f(t, y)$ with the prescribed initial conditions. Determine the order of accuracy and the stability region of the method.