國立臺灣大學95學年度碩士班招生考試試題

Linear Algebra

Instructions: You MUST show your work in order to receive full credit!

1. Let

$$A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}.$$

(a) *(10 points)* Find the characteristic polynomial of A.

(c) *(8 points)* Find a Jordan canonical form of A.

(c) *(7 points)* Find a matrix B that is similar to e^A, where $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

2. *(20 points)* Let V be a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ and let $T : V \to V$ be a linear map. Suppose that all the eigenvalues of T lie in the field. Prove that V has an orthonormal basis with respect to which T is upper triangular, i.e. a matrix that is zero below the diagonal.

3. Let V be a finite-dimensional vector space and $T : V \to V$ be a linear map. Suppose that $W \subseteq V$ is a T-invariant subspace, i.e. $T(W) \subseteq W$. Then T induces linear maps $T|_W : W \to W$ and $\overline{T} : V/W \to V/W$.

(a) *(5 points)* Suppose that $T : V \to V$ is diagonalizable. Prove that the linear maps $T|_W : W \to W$ and $\overline{T} : V/W \to V/W$ are both diagnoalizable.

(b) *(10 points)* Suppose that $T|_W : W \to W$ and $\overline{T} : V/W \to V/W$ are diagonalizable. Assume in addition that $T|_W$ and $\overline{T}$ have no common eigenvalues. Prove that T is diagonalizable.

(c) *(5 points)* Show by an example that the additional assumption in (b) cannot be removed.

4. *(15 points)* Let $Mat_{n\times n}(F)$ be the vector space of $n \times n$ matrices over F. Let $A \in Mat_{n\times n}(F)$ and define a linear map $\mathrm{ad}A : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ by the formula

$$\mathrm{ad}A(X) := AX - XA, \quad X \in Mat_{n\times n}(F).$$

Suppose that A is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$. Prove that $\mathrm{ad}A$ is diagonalizable. What are the eigenvalues of $\mathrm{ad}A$?

5. Let V be a finite-dimensional vector space and $T : V \to V$ be a linear map.

(a) *(15 points)* Prove that there exists a positive integer k such that $V = \mathrm{Ker}T^k \oplus \mathrm{Im}T^k$.

(b) *(5 points)* Show by an example that (a) is not true for infinite-dimensional vector spaces.