1. Suppose (X, Y) has joint probability density function

$$f(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2 + x - y} I_{(x \le y)},$$

where $\sigma > 0$ and $I_{(x\le y)}$ equals 1 if $x \le y$ and 0 otherwise.

(a) (10 pts) Find the mean and variance of Y.

(b) (10 pts) Find $g_0(X)$ and $\mathrm{Var}\{g_0(X)\}$ where g_0 minimizes $\mathrm{E}\{Y - g(X)\}^2$ among all measurable functions g.

(c) (6 pts) Suppose $\{(X_1, Y_1), \cdots, (X_n, Y_n)\}$ is a random sample from $f(x, y)$ given above. Derive the level-α likelihood ratio test for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \ne \sigma_0$, where $\sigma_0 > 0$ is a given constant.

2. Suppose X is uniformly distributed on $\{-1, 1\}$ and Z has probability density function $f_Z(z) = \sigma^{-1} \exp(-z/\sigma) I_{(z\ge 0)}$, where $\sigma > 0$ and $I_{(z\ge 0)}$ equals 1 if $z \ge 0$ and 0 otherwise. For an odd n, let $Y_1, \cdots, Y_n$ be a random sample on $Y = XZ + \mu$, where $-\infty < \mu < \infty$.

(a) (5 pts) Find the method of moments estimator of μ.

(b) (10 pts) Find the maximum likelihood estimators of μ and σ.

(c) (10 pts) Derive the asymptotic distributions of the estimators of μ in (a) and (b) and compare them.

(d) (5 pts) Derive the asymptotic distribution of the estimator of σ in (b).

(e) (8 pts) Derive an approximately level-α test for $H_0 : \mu \le 0$ versus $H_1 : \mu > 0$.

(f) (8 pts) Derive an approximately level-$(1 - \alpha)$ confidence interval for μ.

3. Let $X_1, \cdots, X_n$ be a random sample from a Poisson(λ) distribution with mean λ.

(a) (14 pts) Find a conjugate prior distribution for λ and the corresponding Bayes estimator of λ under square loss.

(b) (8 pts) Derive the level-α uniformly most powerful test for $H_0 : \lambda \ge \lambda_0$ versus $H_1 : \lambda < \lambda_0$, where $\lambda_0 > 0$ is a given constant.

(c) (6 pts) Derive an approximately level-$(1 - \alpha)$ confidence interval for λ.