Linear Algebra

(1至5題爲複選題，每題5%，答案須寫於答案卷首頁選擇題作答區對應位置，全對才給分。)

1. Let $A=\begin{bmatrix}2&0&-1&0\\0&2&0&-1\\-1&0&2&0\\0&-1&0&2\end{bmatrix}$, and $B=PAP^T=\begin{bmatrix}2&-1&0&0\\-1&2&0&0\\0&0&2&-1\\0&0&-1&2\end{bmatrix}$ with P being a permutation matrix. Denote the (i, j)-entry of P as P_{ij}, then (A) $P_{12}P_{22} = 1$. (B) $P_{31} = 1$. (C) $P_{43} = 0$. (D) trace(P) = 2. (E) $P^T = P^{-1}$.

2. Let A be a diagonalizable $n\times n$ matrix with $p(t)=(-1)^n(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)=c_nt^n+c_{n-1}t^{n-1}+\cdots c_0$ as its characteristic polynomial, then (A) $\lambda_1, \lambda_2, \ldots \lambda_n$ must be distinct. (B) trace$(A)=\sum_{k=1}^{n}\lambda_k$. (C) $c_{n-1} = (-1)^{n-1}$ trace(A). (D) c_0 = det(A). (E) There must exist an $n\times n$ matrix C such that $C^3 = A$.

3. Let A be a 5×3 matrix with orthonormal columns, then (A) det(A^TA) = 0. (B) det(A^TA) = 1. (C) det(AA^T) = 1. (D) det(AA^T) = 0. (E) A^T has a non-trivial null space of dimension 2.

4. Consider an $n\times n$ matrix A_n with the (i, j)-entry being $min(i, j)$, then (A) det(A_5) = 1. (B) det(A_{10}) = −1. (C) $\forall\, n > 1$, det(A_n) = n. (D) $\forall\, n > 1$, det(A_n) = det(A_{n-1}). (E) $\forall\, n > 1$, A_n is diagonalizable.

5. Choose one or more TRUE statement(s) from the followings:
   (A) If A and B are invertible $n\times n$ matrices, then AB^T is also invertible.
   (B) If A and B are $n\times n$ matrices such that B is invertible, then det(BAB^{-1}) = det(A).
   (C) For any $m\times n$ matrix A and $n\times p$ matrix B, the null space of B is contained in the null space of AB.
   (D) An $n\times n$ matrix A is diagonalizable if and only if there is a basis for $\mathcal{R}^n$ consisting of eigenvectors of A.
   (E) Distinct eigenvectors of a symmetric matrix are orthogonal.

(6、7兩題爲非選擇題，答案須寫於答案卷非選擇題作答區，並標明題號)

6. Suppose the matrix A has rank r and $A = PR$, where P is an invertible matrix and R is the reduced row echelon form of A. Let Col A be the column space of A, (Row A)$^\perp$ be the orthogonal complement of the row space of A with respect to the dot product, and Null R be the null space of R.
   (a) Prove that the first r columns of P form a basis of Col A. (10%)
   (b) Prove that (Row A)$^\perp$ = Null R. (5%)

7. In the inner product space $\mathcal{R}^3$ with the inner product function $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}^TD\mathbf{v}$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, where $D = [\ \mathbf{e}_1\ \ 4\mathbf{e}_2\ \ \mathbf{e}_3\ ]$ and $\mathbf{e}_i$ is the ith standard vector of $\mathcal{R}^3$, find the least-squares approximation $\mathbf{x}$ of the following problem:

$$\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\mathbf{x}=\begin{bmatrix}-1\\2\\1\end{bmatrix}.\quad (10\%)$$

接背面

Differential Equations

(8、9、10 三題為非選擇題，答案須寫於答案卷非選擇題作答區，並標明題號)

There are two species of animals: wolf and fox, interacting within the same forest ecosystem. Let $w(t)$ and $f(t)$ denote the wolf and fox populations, respectively, at time t. Suppose further that wolves might eat foxes as food and foxes might also eat wolves as food, but only wolves are hunted by human. If there are no foxes, then one might expect that the wolves, lacking an adequate food supply, would decline in number at a rate of $-11w(t)$. When foxes are present, there is a supply of food, and so wolves are added to the forest at a rate of $3f(t)$. Furthermore the changing rate of the wolf population also positively depends on a seasonal hunting factor, denoted as $100\sin(t)$. On the other hand, if there is no wolves, then the foxes, lacking an adequate food supply, would decline in number at a rate of $-3f(t)$. But when wolves are present, the foxes population is increased by a rate of $3w(t)$.

Please answer the following questions:

8. Formulate the above system by a set of differential equations. (5%)

9. Use variation of parameters to solve the system. That is, find the solution for $w(t)$ and $f(t)$. (15%)

10. What are the steady-state pollutions of the wolf and fox, respectively? (5%)

(11 至 15 題為複選題，每題 5%，答案須寫於答案卷首頁選擇題作答區對應位置，全對才給分。)

11. A thermometer reading 70°F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $T(t=0.5) = 110\,^{\circ}$F after 0.5 minute and $T(t=1) = 145\,^{\circ}$F after 1 minute. Find the oven temperature T_{oven} and $T(t=2)$.

(A) $T_{oven} = 370\,^{\circ}$F, (B) $T_{oven} = 380\,^{\circ}$F, (C) $T_{oven} = 390\,^{\circ}$F, (D) $T(t=2) > 200\,^{\circ}$F, (E) $T(t=2) > 210\,^{\circ}$F.

12. Given that $y = \sin x$ is a solution of $y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$, choose the functions which are also solutions of the differential equation.

(A) $\cos 2x$, (B) $e^{-x}\cos 3x$, (C) $\cos x$, (D) $\sin 2x$, (E) $e^{x}\sin 2x$.

13. A 100-volt electromotive force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ as $t \to \infty$, and the current $i(t)$ on the capacitor if $q(0) = 0$.

(A) $q(t \to \infty) = 10^{-4}$, (B) $q(t \to \infty) = 1/100$, (C) $q(t \to \infty) = 1/1000$, (D) $i(t) = 0.5e^{-50t}$, (E) $i(t) = 0.05e^{-50t}$.

14. Note that $x = 0$ is an ordinary point of the differential equation $y'' + x^2 y' + 2xy = 5 - 2x + 10x^3$. Find the P, Q, R, S, and T of the solution:

$$y = c_0(1 + Px^3 + \cdots) + c_1(x + Qx^4 + Rx^7 + \cdots) + Sx^2 + Tx^3 + \cdots$$

(A) $P = \frac{1}{6}$, (B) $Q = -\frac{1}{4}$, (C) $R = \frac{1}{27}$, (D) $S = \frac{5}{2}$, (E) $T = -\frac{1}{3}$.

15. Solve the given equation $\int_0^t f(\tau) f(t-\tau)\,d\tau = 6t^3$,

(A) $f(t) = 6t$, (B) $f(t) = 3\sqrt{2}t$, (C) $f(t) = \sqrt{6}t$, (D) $f(t) = -6t$, (E) $f(t) = -2t$.

試題隨卷繳回