1. (10 points) Determine the maxima and minima of the function

$$f(x,y) = (ax^2 + by^2)\, e^{-x^2-y^2} \qquad (0 < a < b)\ .$$

2. (10 points) Let $Q = [0,1] \times [0,1]$,calculate $\int\int_Q f(x,y)dxdy$ in the following case.
   $f(x,y) = x^2 + y^2$ if $x^2 + y^2 \le 1$, $f(x,y) = 0$ otherwise.

3. (20 points)
   (a) Show $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ converges for all $x \in R$.
   (b) Let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$,show $f(x)$ is differentiable for all $x \in R$.

4. (10 points) Let $F(x,y)$ be continuous on $[a,b] \times [c,d]$, and suppose $\{\varphi_n(x)\}$ converges uniformly on $[a,b]$ with $c \le \varphi_n \le d$. Show that the sequence $f_n(x) = F(x, \varphi_n(x))$ converges uniformly on $[a,b]$.

5. (20 points) Let f be a map from $\mathbb{R}^2 = \{(x,y) : x, y \in R\}$ to $\mathbb{R}$, defined as

$$f(X) = ye^x + xe^y + x + y\ , \qquad here\ \ X = (x,y).$$

   Let $f(X) = f(0,0) + XA + XBX^t + E(X)$, be the quadratic approximation of f at $(0,0)$, here $\lim_{\|X\|\to 0} E(X)/\|X\|^2 = 0$, $\|X\| = \sqrt{x^2+y^2}$. Compute the vector A and the matrix B.

6. (10 points) Consider the transformation

$$\begin{cases} u = \varphi(\xi,\eta) \\ v = \psi(\xi,\eta) \end{cases} \qquad \begin{cases} \xi = f(x) \\ \eta = g(y) \end{cases}$$

   Show that

$$\frac{\partial(u,v)}{\partial(x,y)} = f'(x)\ g'(y)\ \frac{\partial(u,v)}{\partial(\xi,\eta)}$$

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,here $\dfrac{\partial(u,v)}{\partial(x,y)}$ is the Jacobian determinant.

7. (20 points) A transformation in the plane

$$x = \phi(u,v), \qquad y = \psi(u,v)$$

is called conformal if it maps any two intersecting curves into two other curves enclosing the same angle as the original ones.
You may use the following Theorem without proof.

Theorem. A necessary and sufficient condition that a continuously differentiable transformation should be conformal is that the Cauchy-Riemann equations

$$\phi_u - \psi_v = 0 \ , \qquad \phi_v + \psi_u = 0$$

or

$$\phi_u + \psi_v = 0 \ , \qquad \phi_v - \psi_u = 0$$

hold. In the first case the direction of the angles is preserved, in the second case the direction is reversed.

a) Prove that the inversion

$$\xi = \frac{x}{x^2+y^2} \ , \qquad \eta = \frac{y}{x^2+y^2}$$

is a conformal transformation;

b) prove that under this inversion transformation, any circle is another circle or a straight line;