1. (20 pts) Solve the following initial valued problem

$$\begin{aligned} \frac{d}{dt}x &= 2y\,, \\ \frac{d}{dt}y &= 2x+3y\,, \quad t>0\,, \\ x(0) &= 1\,, \quad y(0)=0\,, \end{aligned}$$

where $x=x(t)\in\mathbb{R}$, and $y=y(t)\in\mathbb{R}$.

2. (20 pts) Consider the problem of ordinary differential equation

$$\begin{aligned} \frac{d}{dt}z &= -\tan z\,, \quad z=z(t)\in\mathbb{R}\,, \quad t>0\,, \\ z(t) &\to 0 \quad \text{as} \quad t\to+\infty\,. \end{aligned}$$

Can the problem have a unique solution? Prove or disprove your answer.

3. (20 pts) Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function. Consider the initial valued problem

$$\begin{aligned} \frac{d}{dt}x &= f(x)\,, \quad x=x(t)\in\mathbb{R}\,, \quad t>0\,, \\ x(0) &= x_0\,, \end{aligned}$$

where x_0 is a nonnegative constant.

(i) Must $\sup_{0<t<T}|x(t)|<+\infty$, for $T>0$? (10 pts)

(ii) Assume $f(0)=0$ and $f'(0)>0$. Can the problem of ordinary differential equation

$$\begin{aligned} \frac{d}{dt}y &= f(y)\,, \quad y=y(t)\in\mathbb{R}\,, \quad t>0\,, \\ y(t) &\to 0 \quad \text{as} \quad t\to+\infty\,, \end{aligned}$$

have a unique solution? (10 pts) Prove or disprove all your answers.

4. (20 pts) Let k be a positive integer. Consider the ordinary differential equation (ODE)

$$\frac{d^2}{dt^2}x-2c\frac{d}{dt}x+x=0\,, \quad x=x(t)\in\mathbb{R}\,, \quad t>0\,.$$

Can there exist a real number c such that the ODE has a solution satisfying $x(0)=x(2k\pi)=0$? Prove or disprove your answer.

5. (20 pts) Consider the ODE problem

$$\begin{aligned} &\frac{d^2}{dr^2}u+\frac{1}{r}\frac{d}{dr}u-\frac{1}{r^2}u+(1-u^2)u=0\,, \quad r>0\,, \\ &u(0)=0\,, \quad u=u(r)\geq 0\,. \end{aligned}$$

Answer the following questions:

(1) Can there exist any nontrivial solution which is analytic at $r=0$ i.e. the solution u can be represented as $u(r)=\sum_{n=0}^{\infty}a_n r^n$ for $0<r<r_0$, where a_n's are constants and r_0 is a positive constant. (10 pts)

(2) Is there any solution which is increasing to r? Hint: change of variable $r=e^t$. (5 pts)

(3) Is there any solution which tends to 1 as r goes to infinity? (5 pts)

Prove or disprove all your answers.