Linear Algebra

(1) (25 points) Show that if $A \in M_n(\mathbb{R})$ with $A^t = A$ then A is diagonalizable. Here A^t denotes the transpose of A.

(2) Let

$$X = \{(x, y, 1) \mid x, y \in \mathbb{R} \text{ and } 2x^2 + y^2 = 1\},$$

and consider

$$Y = \{Q \in \mathbb{R}^3 \mid Q = tP \text{ for some } t \in \mathbb{R}, P \in X\}.$$

(a) (10 points) Show that the intersection $Y \cap P$, where P is the plane defined by the equation $y - z = 3$, is a parabola.

(b) (10 points) Show that if L is a plane in $\mathbb{R}^3$, then $Y \cap L$ is always a conic (parabola, ellipse, or hyperbola).

(3) (15 points) Let $f(x) = (x - a_1)(x - a_2)\cdots(x - a_{10})$, where $a_1, \ldots, a_{10} \in \mathbb{R}$ are distinct real numbers. For each $v = (b_1, \ldots, b_{10}) \in \mathbb{R}^{10}$, let $g(x) = c_9x^9 + c_8x^8 + \ldots + c_1x + c_0$ be such that

$$\frac{b_1}{x - a_1} + \frac{b_2}{x - a_2} + \ldots \frac{b_{10}}{x - a_{10}} = \frac{g(x)}{f(x)},$$

and define $\Phi(v) = (c_0, c_1, \ldots, c_9)$. Show that Φ is a bi-jection between $\mathbb{R}^{10}$ and $\mathbb{R}^{10}$.

(4) (15 points) Let $v_1 = (a_{1,1}, \ldots, a_{1,6})$, $v_2 = (a_{2,1}, \ldots, a_{2,6}) \in \mathbb{R}^6$ and let

$$P = \{sv_1 + tv_2 \mid 0 \le s, t \le 1\} \subset \mathbb{R}^6.$$

Prove that the area of P equals $\sqrt{A \cdot A^t}$, where $A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,6} \\ a_{2,1} & \cdots & a_{2,6} \end{bmatrix}$ and A^t is the transpose of A.

(5) Let

$$\sum = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \text{ and } \bigwedge = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) (5 points) Show that $\sum \bigwedge = \bigwedge \sum$.

(b) (10 points) Find all the eigen-values and the corresponding eigen-spaces (in the complex vector space $\mathbb{C}^6$) of $\bigwedge$.

(c) (10 points) Find all the eigen-values and the corresponding eigen-spaces (in $\mathbb{C}^6$) of $\sum$.