國立臺灣大學97學年度碩士班招生考試試題

(1) (15%) Prove that $(p-1)! \equiv p-1 \pmod{1+2+\cdots+(p-1)}$ if p is a prime number.

(2) (15%) Let K be an extension field of F and let $A \in M_n(K)$. the set of $n \times n$ matrices with entries in K. Show that A can be written as $A = \lambda_1 A_1 + \cdots + \lambda_k A_k$ such that (i) $\lambda_i \in K$. $A_i \in M_n(F)$ for all i: and (ii) if $C \in M_n(F)$ commutes with A. then C commutes with each A_i.

(3) (15%) Let G be a group and H a subgroup of finite index n. Prove that there exists a normal subgroup of G contained in H and has index dividing $n!$.

(4) (15%) (a) Let G be a finite abelian group. Let d be the product of all elements of G. Prove that $d^2 = 1$.

(b) Let F be a finite field. Let d be the product of all nonzero elements. Prove that $d = -1$.

(5) (20%) Let $R = M_n(\mathbb{Z})$ be the ring of all $n \times n$ matrices with integer entries. A set I is a left ideal if it is an additive subgroup of R and for any $r \in R$ and $a \in I$. we have $ra \in I$. Classify all left ideals of R.

(6) (20%) Let R be a commutative ring. An element $a \in R$ is called a zero divisor if there exists $b \neq 0$ such that $ab = 0$. Suppose R has only a finite number $n(>1)$ of zero divisors.

(a) Let $a \neq 0$ be a zero divisor. Show that the set $I = \{r : ra = 0\}$ has at most n elements.

(b) Prove that R has at most n^2 elements.

(c) Give an example: A ring with p^2 elements and the number of zero divisors is p. where p is a prime number.

試題隨卷繳回