1. (a) (10%) Solve the differential equation

$$\frac{dy}{dt} = y(y-\alpha)(1-y),\ 0<\alpha<1.$$

You may express the solutions in implicit form.

(b) (10%) Show that for any initial datum $y(0)=y_0$, the corresponding solution $y(t)\to 1$ or 0, as $t\to\infty$ depending on $y_0>\alpha$ or $y_0<\alpha$.

2. (a) (10 points) Find general solutions of the forced damping oscillator:

$$y'' + \gamma y' + \omega_0^2 y = A\sin\omega t$$

where $\gamma\geq 0, \omega_0\geq 0, \omega>0, A>0$ are constants.

(b) (5%) If $\gamma>0$, what is the corresponding asymptotic solution? Show that general solutions tend to the asymptotic solution as $t\to\infty$.

(c) (5%) If $\gamma=0$ and $\omega=\omega_0$, what is the corresponding solution? Show that the solution $|y(t)|\to\infty$ as $t\to\infty$.

3. For functions defined on $(0,\infty)$ and satisfying $|y(t)|\leq Me^{at}$ for some positive constants M and a, define the Laplace transform

$$L(y)(s) := \int_0^\infty e^{-st}y(t)\,dt,\ Re(s)>a,$$

and the convolution operation:

$$(f*g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau,\ t>0.$$

(a) (10%) Show that $L(f*g)=L(f)\,L(g)$.

(b) (10%) Consider a differential operator $P(D) := (D^2+\omega^2)$, where $D=d/dt$, $\omega>0$ is a constant. The solution corresponding to

$$P(D)y=f,\ \text{with } y(0)=y'(0)=0$$

can be expressed as $y(t)=(G*f)(t)$. Find explicit expression of G.

4. Find the stationary points (equilibria) of the following predator-prey system

$$x' = x(y-1),\ y' = ry(1-x),$$

where $r>0$ is a constant.

(a) (5%) Find the equilibria (stationary points).

(b) (5%) Classify the qualitative behaviors (sink, source, saddle,spiral,...) of the equilibria.

(c) (10%) Sketch the solution structure on the x-y plane in the first quadrant (i.e. equilibria, nullclines, vector field, trajectories).

5. Consider the linear equation:

$$\begin{cases} x' = 3x-4y+f(t) \\ y' = x-y \end{cases}$$

(a) (10%) When $f(t)\equiv 0$, find the general solutions.

(b) (10%) When $f(t)=e^t$, what is the corresponding general solutions?