第一大題為選擇題，共有 16 題。考生應以 2B 鉛筆作答於「答案卡」，未依規定作答於答案卡者，本大題不予計分。

單選題(1 至 6 題，每題 5 分)

1. Given the linear operator T with standard matrix $[T]_E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ and

B-matrix $[T]_B = \begin{bmatrix} 1 & 9 & -6 \\ 0 & 7 & -4 \\ 2 & 11 & -8 \end{bmatrix}$, which can be a correct basis for B?

(A) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 9 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

(D) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix} \right\}$

(E) None of the above.

2. The commutator of two $n \times n$ matrices A and B is defined as $[A, B] = AB - BA$. Let $\underline{0}$ denote the $n \times n$ zero matrix. For $n \times n$ matrices A, B, and C, which of the following statements is NOT correct?

(A) $[A, B] = -[B, A]$.

(B) $[A, B+C] = [A, B] + [A, C]$.

(C) $[A, BC] = [A, B]C + B[A,C]$.

(D) $[A, [B, C]] + [B, [A, C]] + [C, [A, B]] = \underline{0}$.

(E) If $[A, B] = \underline{0}$ and $[B, C] = \underline{0}$, then $[A, C] = \underline{0}$.

3. An $n\times n$ matrix A is called diagonalizable if $A=PDP^{-1}$ for some diagonal $n\times n$ matrix D and some invertible $n\times n$ matrix P. Choose the following matrix which is diagonalizable.

(A) $\begin{bmatrix}-1&0&0\\-4&-2&5\\-4&-5&8\end{bmatrix}$ (B) $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ (C) $\begin{bmatrix}3&0&0\\0&1&2\\0&2&1\end{bmatrix}$ (D) $\begin{bmatrix}5&5&-6\\0&-1&0\\3&2&-4\end{bmatrix}$

(E) $\begin{bmatrix}5&0&0\\0&0&-2\\0&2&0\end{bmatrix}$

4. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v}=0$. The $S^{\perp}$ is the set of all vectors in R^n that are orthogonal to every vector in S. Consider the set $S=\left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in R^3 : x_1-x_2+x_3=0\right\}$.

Choose the following statement which is correct.

(A) $\begin{bmatrix}1\\1\\0\end{bmatrix}\in S^{\perp}$ (B) $\begin{bmatrix}1\\-1\\1\end{bmatrix}\in S$

(C) S is a subspace of R^3 and $dim\ S=1$.

(D) Let $\begin{bmatrix}1\\0\\0\end{bmatrix}=\mathbf{w}+\mathbf{z}$ such that $\mathbf{w}\in S$ and $\mathbf{z}\in S^{\perp}$, then $\mathbf{z}=\begin{bmatrix}\frac{1}{3}\\\frac{-1}{3}\\\frac{1}{3}\end{bmatrix}$.

(E) $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}$ is a basis for S.

5. The integrating factor of $a(x)y'(x)=b(x)y(x)+f(x)$ is

(A) $e^{\int b(x)dx}$, (B) $e^{-\int\frac{b(x)}{a(x)}dx}$, (C) $e^{-\int f(x)dx}$, (D) $e^{\int\frac{f(x)}{a(x)}dx}$, (E) $e^{\int\frac{b(x)}{a(x)}dx}$

6. What is the solution of $f(t)$? (hint: using the Laplace transform)

$$\int_0^t e^{\tau}\sin(t-\tau)d\tau=\int_0^t f(\tau)d\tau$$

(A) $\frac{1}{2}e^t\sin(t)+\frac{1}{\sqrt{2}}e^t\cos(t)$, (B) $\frac{1}{\sqrt{2}}\sin(t)+\frac{1}{\sqrt{2}}\cos(t)$, (C) $\frac{1}{4}e^t+\frac{1}{2}\cos\left(t+\frac{\pi}{4}\right)$,

(D) $\frac{1}{\sqrt{2}}e^t\cos\left(t+\frac{\pi}{4}\right)$, (E) $\frac{1}{2}e^t+\frac{1}{\sqrt{2}}\sin\left(t-\frac{\pi}{4}\right)$

複選題(7 至 16 題，每題 5 分。完全答對才計分，不倒扣)

7. Which of the following statements are correct?

(A) For an $n\times n$ matrix A, the columns of A are linearly independent if and only if the rows of A are linearly independent.

(B) For an $m\times n$ matrix A, the nullity of A equals the nullity of its transpose A^T.

(C) An $m\times n$ matrix A defines some linear transformation T_A: $R^n\rightarrow R^m$. T_A is onto if and only if rank $A=m$.

(D) A set S of vectors forms a basis for a subspace V of R^n if and only if the vectors of S are linearly independent and the number of vectors in S equals the dimension of V.

(E) The set $V=\left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in R^3:3x_1+2x_2-x_3=1\right\}$ is not a subspace of R^3.

8. Suppose the matrix $\begin{bmatrix}1&2&3&1&b\\2&5&3&a&0\\1&0&8&6&c\end{bmatrix}$ can be transformed to the reduced row echelon form $\begin{bmatrix}1&0&0&-2&0\\0&1&0&d&-1\\0&0&1&1&e\end{bmatrix}$. Which of the following equalities are correct?

(A) $a=1$. (B) $b=3$. (C) $c=40/3$. (D) $d=-1$. (E) $e=2$.

9. An affine transformation of R^2 is a function $T: R^2 \to R^2$ of the form $T(\bar{x}) = A\bar{x} + \bar{b}$, where A is an invertible 2×2 matrix and $\bar{b} \in R^2$. Which of the following statements are correct?

(A) $T^{-1}(\bar{x}) = A^{-1}\bar{x} - A^{-1}\bar{b}$.

(B) Affine transformations map straight lines to straight lines.

(C) There is no affine transformation that can map a straight line to a circle.

(D) Affine transformations map parallel straight lines to parallel straight lines.

(E) There exists an affine transformation that maps parallel straight lines to intersecting straight lines.

10. Define the linear operator T on R^2 by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix}$. Which statements in the following are correct?

(A) 3 is an eigenvalue of T.

(B) 4 is an eigenvalue of T.

(C) -2 is an eigenvalue of T.

(D) $\left\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$ is a basis for the eigenspace of T.

(E) $\left\{\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right\}$ is a basis for the eigenspace of T.

11. A subset of R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. An orthogonal projection of **v** onto a subspace **W** is defined as a vector, $\mathbf{w} \in \mathbf{W}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{z}$, where $\mathbf{z} \in \mathbf{W}^{\perp}$. Which statements in the following are correct?

(A) Any orthogonal set of nonzero vectors is linearly independent.

(B) Every subspace has an orthogonal basis.

(C) For any matrix A, $(Row\ A)^{\perp} = Null\ A$.

(D) Let **W** be a subspace of R^n and **v** be a vector in R^n. Among all vectors in **W**, the vector closest to **v** is the orthogonal projection of **v** onto $\mathbf{W}^{\perp}$.

(E) For any subspace **W** of R^n, $dim\ \mathbf{W} + dim\ \mathbf{W}^{\perp} = n$.

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12. Let $F(R)$ denote the set of all functions from R to R. Choose the following subsets of $F(R)$ which are linearly independent.

(A) $\{t^2-2t+5, 2t^2-4t+10\}$ (B) $\{\sin t, \sin^2 t, \cos^2 t, 1\}$ (C) $\{t^2-2t+5, 2t^2-5t+10, t^2\}$

(D) $\{t, t\sin t\}$ (E) $\{e^t, e^{2t}, \cdots, e^{nt}, \cdots\}$

13. Consider the equation $\ddot{x}(t)+16x(t)=0$

(A) There are infinite many solutions.

(B) There are no solutions.

(C) There are two independent solutions.

(D) There are no solutions for $x(0)=0$, and $x(\frac{\pi}{2})=0$.

(E) There are infinite many solutions for $x(0)=0$, and $x(\frac{\pi}{2})=1$.

14. Consider $\dot{X}(t)=AX(t)$, where A is an n by n matrix, $X(t)$ is an n by 1 vector, $n\geq 2$

(A) e^{At} = Inverse Laplace transform of $(sI-A)^{-1}$

(B) $X(t)=e^{At}C=Ce^{At}$ for any n by 1 vector C

(C) $(sI-A)$ is nonsingular for any scalar s

(D) e^{At} is nonsingular for any scalar t

(E) $\frac{d}{dt}e^{At}=Ae^{At}=e^{At}A$.

15. What are the function sets listed as follows orthogonal on the interval [0, 1]?

(A) $\{1, \cos 2\pi x, \cos 4\pi x, \cos 6\pi x, \ldots\ldots\ldots\}$

(B) $\{1, x, x^2, x^3, \ldots\ldots\}$

(C) $\{1, \sin 4\pi x, \sin 8\pi x, \sin 12\pi x, \ldots\ldots\ldots\}$

(D) $\{P_0(2x-1), P_1(2x-1), P_2(2x-1), P_3(2x-1), \ldots\ldots\}$, where $P_n(x)$ means the Legendre polynomial.

(E) $\{I_0(x), I_1(x), I_2(x), I_3(x), \ldots\ldots\}$, where $I_\nu(x)$ is the modified Bessel function of the first kind.

16. Suppose that $f(x) = 0$ for $0 < x < 1$, $f(x) = -x + 3$ for $1 < x < 3$,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{2}x, \qquad a_0 = \int_0^2 f(x)\,dx$$

What statements in the following are correct?

(A) $g(x) = g(-x)$, (B) $g(1) = 1$, (C) $g(x) = g(x+2)$, (D) $g(-3/2) = 0$, (E), $g(7/2) = 0$.

第二大題為計算題，共有 2 題，每題 10 分。考生應於試卷上「非選擇題作答區」註明題號，依序作答。

17. Solve $y(x)$

$$y''(x) + 2y'(x) + y(x) = e^{-x} \quad \text{with} \quad y(0) = y'(0) = 1$$ (10 points)

18. Use separation of variables to find the product solutions for the following partial differential equation.

$$x\frac{\partial u(x,y)}{\partial x} = y\frac{\partial u(x,t)}{\partial y}.$$ (10 points)