※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

1. (25 pts) Suppose that (X, Y) have joint probability density function (pdf)

$$f_{XY} = \begin{cases} \sqrt{\frac{2}{\pi}}x \exp\left[-\frac{1}{2}(y-2x^2)^2\right], & \text{if } 0 \le x \le 1; y \in R \\ 0, & \text{otherwise} \end{cases}.$$

 (a) (10 pts) Find $f_X(x)$ which is the marginal probability density function of X.

 (b) (5 pts) Show that the conditional distributional of Y, given $X = x$, is normal with mean $2x^2$ and variance 1.

 (c) (5 pts) Find $E(Y)$.

 (d) (5 pts) Find $Var(Y)$.

2. (25 pts) Suppose that X_n is a random variable having the binomial distribution $Bin(n, p)$, where $0 < p < 1$, $n = 1, 2, \ldots$. Define

$$Y_n = \begin{cases} \log(X_n/n), & X_n \ge 1 \\ 1, & X_n = 0. \end{cases}$$

 (a) (10 pts) Show that Y_n converges to $\log p$ in probability. State your reasoning clearly.

 (b) (15 pts) Show that $\sqrt{n}(Y_n - \log p)$ converges to $N(0, (1-p)/p)$.

3. (25 pts) Let $(Y_1, Z_1), \ldots, (Y_n; Z_n)$ be independent identically distributed with the pdf

$$\lambda^{-1}\mu^{-1}e^{-y/\lambda}e^{-z/\mu}I_{(0,\infty)}(y)I_{(0,\infty)}(z),$$

where $\lambda > 0$ and $\mu > 0$.

 (a) (10 pts) Find the MLE of (λ, μ).

 (b) (7 pts) Suppose that we only observe $X_i = \min(Y_i, Z_i)$ and $\delta_i = 1$ if $X_i = Y_i$ and $\delta_i = 0$ if $X_i = Z_i$. Derive the distribution of X_1 and δ_i.

 (c) (8 pts) Under the setting of (b), find the MLE of (λ, μ).

4. (25 pts) A random sample $X_1, \ldots, X_n$ is drawn from a normal distribution with mean θ and unknown variance σ^2.

 (a) (10 pts) Derive the likelihood ratio statistic Λ for testing the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_a : \theta \ne \theta_0$.
 i.e. The likelihood ratio Λ is defined as the ratio of $\sup\{L((\theta, \sigma^2)|data) : \theta = \theta_0\}$ and $\sup\{L((\theta, \sigma^2)|data)\}$ where L denote the likelihood function.

 (b) (5 pts) Show that Λ is a monotone function of the ratio

$$T = \frac{\sqrt{n(n-1)}(\bar{X} - \theta_0)}{\sqrt{\Sigma_{i=1}^{n}(X_i - \bar{X})^2}}.$$

 (c) (10 pts) What is the distribution of the statistic T when $\theta = \theta_0$? Justify your answer.

試題隨卷繳回