國立臺灣大學九十二學年度轉學生入學考試試題

科目： 微積分(A)

題號：19

共 1 頁之第 全 頁

1. (**25%**) Let $u : [a,b] \to \mathbf{R}$ be a continuous function and $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f''(x) \geq 0$ for all $x \in \mathbf{R}$.

   (a) Prove the **mean value theorem for definite integrals** for u: There exists a $\xi \in [a,b]$ such that $u(\xi) = \dfrac{1}{b-a}\displaystyle\int_a^b u(t)\,dt$.

   (b) Use Taylor expansion to prove that

   $$\frac{1}{b-a}\int_a^b f(u(t))\,dt \geq f\Big(\frac{1}{b-a}\int_a^b u(t)\,dt\Big).$$

   When does "=" hold in the above inequality?

2. (**25%**) Let $f : [0,\infty) \to \mathbf{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = L$. Let $b > a > 0$ be two arbitrary positive numbers.

   (a) Explain why $\displaystyle\int_0^\infty \frac{f(bx)-f(ax)}{x}\,dx$ is an improper integral and how it is defined.

   (b) Show that the improper integral $\displaystyle\int_0^\infty \frac{f(bx)-f(ax)}{x}\,dx$ converges and has the value $(L - f(0))\ln(b/a)$.

   (c) Evaluate

   $$\int_0^\infty \frac{\tan^{-1}(\pi x) - \tan^{-1}(2x)}{x}\,dx.$$

3. (**24%**) Briefly describe the **geometric meaning** of the following statements or quantities. Here $f : \mathbf{R}^n \to \mathbf{R}$ is a scalar function on $\mathbf{R}^n, n \geq 2$.

   (a) f is **differentiable** at $(a_1, a_2, \ldots, a_n)$.

   (b) The **gradient** $(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ of a differentiable function f, where $f_{x_j} = \partial f/\partial x_j$.

   (c) The **Jacobian** $\dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}$ of the transformation T given by $x_1 = x_1(u_1, u_2, \ldots, u_n), \ldots, x_n = x_n(u_1, u_2, \ldots, u_n)$.

4. (**26%**) (a) Evaluate the integral

   $$\iint_{\mathcal{R}} e^{xy - x^2 - y^2}\,dA, \qquad \mathcal{R} = \{(x,y) \in \mathbf{R}^2, x^2 - xy + y^2 \leq 5\}.$$

   (b) Consider the following iterated integral

   $$\int_0^4 \Big(\int_{y^{3/2}}^8 y^2 \sin(x^3)\,dx\Big)\,dy.$$

   Graph the region $\mathcal{D}$ of integration and then evaluate the integral.

試題必須隨卷繳回