[Answers in both Chinese and English are OK]

1.(12%) Find the dimension of, and a basis for, the solution space of each of the following homogeneous systems:

(a) $\begin{aligned} x_1+3x_2+2x_3 &= 0 \\ x_1+5x_2+x_3 &= 0 \\ 3x_1+5x_2+8x_3 &= 0 \end{aligned}$; (b) $\begin{aligned} 3x_1+x_2+x_3+x_4 &= 0 \\ 5x_1-x_2+x_3-x_4 &= 0 \end{aligned}$.

2.(18%) One description of spin 1 particles in quantum mechanics uses the matrices

$$M_x=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_y=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \text{ and } M_z=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Are they Hermitian matrices?

Show that

(b) $[M_x, M_y]=iM_z$, where $[A,B]=AB-BA$ is called commutator;

(c) $M^2 \equiv M_x^2+M_y^2+M_z^2=2I$;

(d) $[M^2, M_y]=0$.

If $L^{\pm}=M_x \pm iM_y$, then

(e) $[L^+, L^-]=2M_z$;

(f) $[M_z, L^-]=-L^-$.

3.(12%) Given a matrix $A=\begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where α and β are non-zero complex numbers, find its eigenvalues and eigenvectors. Find the respective conditions for (a) the eigenvalues to be real and (b) the eigenvectors to be orthogonal. Show that the conditions are jointly satisfied if and only if A is Hermitian.

4.(12%) Find the condition(s) on α such that the simultaneous equations

$$\begin{aligned} x_1 + \alpha x_2 \quad\quad &= 1 \\ x_1 - x_2 + 3x_3 &= -1 \\ 2x_1 - 2x_2 + \alpha x_3 &= -2 \end{aligned}$$

have (a) exactly one solution, (b) no solutions, or (c) an infinite number of solutions; give all solutions where they exist.

5.(12%) Consider the polynomial space $P_2(R)$ with the inner product $<f(x), g(x)>=\int_{-1}^{1} f(t)g(t)dt$. The standard basis β is $\{1, x, x^2\}$. Find an orthonormal basis for $P_2(R)$ from β.

6.(12%) Solve the exact differential equation $y(2x^2y^2+1)y'+x(y^4+1)=0$.

7.(12%) For a damped oscillator ($\gamma<\omega_0$), the displacement $x(t)$ at time t satisfies the equation

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0.$$

Find the general solution of this differential equation.

8.(10%) Find, in the form of an integral, the solution of the differential equation

$$\alpha\frac{dy}{dt}+y=f(t)$$

for a general function $f(t)$. Find the specific solutions for

(a) $f(t)=H(t)$, where $H(t)=1$ for $t\geq 0$, and $=0$ for $t<0$.

(b) $f(t)=\delta(t)$,

(c) $f(t)=\beta^{-1}e^{-t/\beta}H(t)$ with $\beta<\alpha$.

For case (c), what happens if $\beta \to 0$?