1. (30%) Write down on the answer sheet the correct answer to each of the following questions. (Derivations are not required.) **(本題請於答案卷之「選擇題作答區」內作答)**

(1) The directional derivative of the scalar function $\varphi(x,y,z)=xy-z^2$ evaluated at point $(1,-1,1)$ along the direction $3\,\underline{\mathbf{i}}-4\,\underline{\mathbf{k}}$ is

(a) 5; (b) 1; (c) -11/5; (d) 11/5; (e) 11.

(2) The line integral $\oint_C \underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ of the 2-D vector function $\underline{\mathbf{F}}=\dfrac{-y}{x^2+y^2}\,\underline{\mathbf{i}}+\dfrac{x}{x^2+y^2}\,\underline{\mathbf{j}}$ evaluated over the closed path C: $(x-2)^2+(y-2)^2=9$ is

(a) π; (b) 0; (c) 2π; (d) 6π; (e) $\pi/2$.

(3) Let $\underline{\mathbf{F}}(r,\theta)=(-1/r)\,\underline{\mathbf{e}}_r$ be a 2-D vector function given in terms of polar coordinates (r,θ) with $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_\theta$ denoting the base vectors of the coordinate system, then $\nabla\cdot\underline{\mathbf{F}}=?$

(a) 0; (b) $-1/r^2$; (c) $1/r^2$; (d) $1/r^3$; (e) $2/r^2$.

Let $f(x)=a_o+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$ be the Fourier series representation of the function $f(x)$ over the interval $-L\le x\le L$. Answer questions (4)~(6).

(4) Which of the following statements regarding to the above Fourier series is true?

(a) $a_o=\frac{1}{L}\int_{-L}^{L}f(x)\,dx$; (b) If $f(x)$ is an odd function in [-L, L], then $b_n=0$ for all n;

(c) $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$; (d) $a_n=\frac{2}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx$;

(e) $\frac{1}{2L}\int_{-L}^{L}f^2(x)\,dx=a_o^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

(5) If in the interval [-2, 2], $f(x)$ is defined as $\begin{bmatrix} f(x)=5 & x=-2 \\ =-x & -2<x\le 0 \\ =x^2-1 & 0<x\le 2\end{bmatrix}$, then the Fourier series at $x=2$ converges to

(a) 7/2; (b) 3; (c) 0; (d) 4; (e) 5/2.

(6) Let $f(x)=\cos^2(\pi x/2)$ in the interval [-2, 2], then which of the following statements is true?

(a) $a_n=0$ for all n; (b) $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=1/4$; (c) $a_o=1/4$; (d) $a_1=1$; (e) $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=1/2$.

(7) Let $\delta(t)$ denote Dirac delta function and $f(t)=\cos t$, then the Fourier transform of $\delta(t-2)f(t)$ is : (note that Fourier transform is defined as $\Im\{g(t)\}=\int_{-\infty}^{\infty}g(t)e^{-i\omega t}dt$)

(a) 0; (b) $e^{-2i\omega}$; (c) $e^{-i(\omega+2)}/\omega$; (d) $e^{-2i\omega}\cos 2$; (e) 1.

(8) Let $z=x+iy$ denote complex variable, then which of the following statements is true?

(a) $f(z)=\ln z$ is an analytic function in $-2\pi\le\arg(z)\le 2\pi$; (b) $z^3+i=0$ has infinitely many complex roots; (c) $f(z)=\sqrt{z}$ has a Taylor series expansion about $z=0$; (d) $f(z)=(1-\cos z)/z$ has a simple pole at $z=0$; (e) $\oint_C dz/[z(z^2+4)]=0$ over the closed circle C: $|z+3|=2$.

接背面

(9) The residue of the complex function $f(z) = \sin z / [z(z+i)^2]$ at $z = -i$ is
(a) $i\cos i - \sin i$; (b) $-i\sin i$; (c) $-i\cos i + \sin i$; (d) 0; (e) $(i\cos i - \sin i)/2$.

(10) What is the value of the complex integral $\oint_C z e^{1/z} dz$ over C: $|z| = 2$?
(a) 0; (b) $2\pi i$; (c) $4\pi i$; (d) πi; (e) 2π.

2. (10%) Consider the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

subject to boundary conditions: $u(0,t) = u(1,t) = 0 \quad (t > 0)$
and initial condition: $u(x,0) = \sin(\pi x)\cos(\pi x) \quad (0 \le x \le 1)$
Solve the problem by using the method of separation of variables. **<u>(Other methods are not allowed.)</u>**

3. (15%) Find the positive eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem:
$y'' + (1+\lambda) y = 0, \quad y(0) + y'(0) = 0, \quad y(\pi) + y'(\pi) = 0$

4. (15%) Find the general solution of $\dfrac{dy}{dx} = \dfrac{2x^2 - y}{x \ln(x)}$.

5. (15%) Answer the following questions.
(1) Consider a set V, consisting of all the real solution functions $y(x)$ of the ordinary differential equation:
$\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$. Is V a real linear vector space? If yes, find the dimension and a basis of the vector space V.
(2) The vector v has components (1,-2,-1) with respect to the basis {(1,-1,1),(1,1,0),(1,0,1)} of R^3. Find its components with respect to the standard basis {(1,0,0),(0,1,0),(0,0,1)}.
(3) Which set or sets of the following vectors can form a basis for R^3? (a), (b), (c), and/or (d)?
(a) (1,2,-1) and (0,3,1)
(b) (2,4,-3), (0,1,1), and (0,1,-1)
(c) (1,5,-6), (2,1,8), (3,-1,4), and (2,1,1)
(d) (1,3,-4), (1,4,-3), and (2,3,-11)

6. (15%) Consider the following system of ordinary differential equations

$$\frac{du}{dt} = Au \text{ where } u = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \in R^3 \text{ and } A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(1) Find the eigenvalues and the associated eigenvectors of matrix A.
(2) Find the exponential of the matrix At.
(3) Find the general solution of the system of ordinary differential equations.

試題隨卷繳回