1 . (20%) Suppose that n people have their hats returned at random. Let $X_i = 1$ if the ith person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the total number of people who get their own hats back. Please calculate the followings

(a) $E(X_i^2)$ (5%)

(b) $E(X_i \bullet X_j)$ for $i \neq j$. (5%)

(c) $E(S_n^2)$. (5%)

(d) $V(S_n)$. (5%).

2. (20%) Let U, V be random numbers chosen independently from the interval [0; 1] with uniform distribution. Find the cumulative distribution and density of each of the variables

(a) $Y = U + V$.

(b) $Y = |U - V|$

3. (10%) Prove that if A and B are independent so are A and $\tilde{B}$.

4. (10%) Find T such that $TH = F$, where

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$$

5. (20%) Let T be the linear operator on R^3 defined by

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ a_2 - a_1 \\ a_3 \end{pmatrix}.$$

(a) What are the nonzero eigenvalues of T?
(b) Find the eigenspace of T corresponding to each nonzero eigenvalue.

6. (20%) Label the following statements as being true or false. (No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%):

(a) Every vector space has a finite basis.
(b) Let V be an inner product space and S be a subset of V. Then $S \subseteq (S^{\perp})^{\perp}$.
(c) If A is invertible, then A^t is also invertible and $(A^t)^{-1} = A^{-1}$.
(d) Let $T, U : V \to W$ be linear transformations. Then $R(T+U) \subseteq R(T) + R(U)$.
(e) Any system of n linear equations in n unknowns has at least one solution.
(f) If both rows of a 2×2 matrix A are identical, then $\det(A) = 0$.
(g) Eigenvectors corresponding to the same eigenvalue are always linearly dependent.
(h) An inner product is a scalar-valued function on the set of ordered pairs of vectors.
(i) If $< x, y >= 0$ for all x in an inner product space, then $y = 0$.
(j) Every orthonormal set is linearly independent.