工程數學 K: 微分方程

1. Consider the following general solution set of a certain differential equation

$$\left\{ y \,\middle|\, y = c_1 x^2 + c_2 e^x, x \in R, c_1 \text{ and } c_2 \text{ are arbitrary real constants} \right\}.$$

(a) If the associated differential equation can be written as $Ly = f(x)$ where $L = \frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$, please find $a_1(x)$, $a_0(x)$ and $f(x)$. (10%)

(b) Find the possible intervals of solutions where the differential equation subject to the initial conditions that $y(x_0) = y_0$, $y'(x_0) = y_1$ where x_0 is inside the interval and both of y_0 and y_1 are arbitrary, has a unique solution. (5%)

(c) If $x_0 = 2$ and there are solutions, what is the relationship between y_0 and y_1? And how many solutions are there? (5%)

(d) For the following new differential equation

$$Ly = x - 2,$$

please find the general solutions. (10%)

2. Find the general solution of the following system of the differential equations

$$\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + 3y + 11z \\ \dfrac{dy}{dt} + \dfrac{dz}{dt} = 2y + 7z \\ \dfrac{dz}{dt} + \dfrac{dx}{dt} = 2x + y + 8z \end{cases}$$

(10%)

3. Consider the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the following boundary conditions

$u(0, y) = 0$, and $u(L, y) = 0$ for $y > 0$

and $u(x, 0) = V$, for $0 < x < L$

where L and V are constants. The possible methods to find the solutions are Fourier series, Fourier complex transform, Fourier cosine transform, Fourier sine transform and Laplace transform. Please choose **two methods** to solve the partial differential equation. You only have to **explain the solution procedures of both methods** in details and indeed **find the solutions using one of them**. (10%)

接背面

工程數學 K: 機率

4. In a lottery game, there are 49 balls, number from 1 to 49. Draw 3 balls without replacement from these 49 balls randomly. Arrange these three numbers in increasing order. (依照球面數字由小而大排列) Take the last digit of each of the three numbers and make a 3-digit number. (Most significant digit from the first number, etc.)

(a) What is the range of this 3-digit number? (5%)

(b) Will this three-digit number have a uniform probability density function? Explain your answer (5%)

(c) Repeat the experiments with 50 balls number from 1 to 50, will the resulting 3-digit number have a uniform probability density function? Explain your answer. (5%)

5. Customers arrive at a bank at a Poisson rate of one per minute. The security guard sitting at the bank's door went out for 10 minutes.

(a) What is the probability that there are exactly 7 customers who arrived at the bank during that interval? (5%)

(b) What are the most likely numbers of customers who arrived during that interval? Why? (Note there can be more than one such number) (5%)

6. A plane is ruled with infinite number of parallel lines with distance d apart. A needle of length l, $l < d$, is tossed at random onto the plane. Let **X** denote the distance from the center of the needle to the closest line, and θ denote the angle between the needle and the line.

(a) What is the joint probability density function f of **X** and θ? **No score would be given unless the formula of f is derived.** (6%)

(b) What is the probability that the needle intersects one of the parallel lines? **No score would be given unless the calculation procedure is shown.** (6%)

7. A set of 200 people, consisting of 100 men and 100 women, is randomly divided into 100 pairs of 2 persons in each pair. Let $\mathbf{X}_i$ be a random variable whose value is 1 when man i is paired with a woman, otherwise its value is zero. **No score would be given unless the calculation procedure is shown.**

(a) What is $E[X_i]$? (6%)

(b) What is $E[X_iX_j]$ for $i \neq j$. (7%)