1. Consider the matrix [20 points]

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(a) Is A a Hermitian matrix? What can one say about the eigenvalues and eigenvectors of a Hermitian matrix?

(b) Find the eigenvalues and eigenvectors of the matrix A.

(c) Given that α is a real parameter, what are the eigenvalues and eigenvectors of $\exp(\alpha A)$, an exponential function of αA defined by $\exp(\alpha A) = \sum_0^\infty \frac{(\alpha A)^k}{k!}$?

2. Given the vector $\vec{A} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$. [20 points]

(a) Evaluate the surface integral $\int (\nabla \times \vec{A}) \cdot d\vec{\sigma}$ over a rectangle in the (x, y) plane bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$. Here, $d\vec{\sigma}$ is a surface element vector.

(b) Evaluate the line integral $\oint \vec{A} \cdot d\vec{\ell}$ around the boundary of the rectangle. Here, $d\vec{\ell}$ is a line element vector.

(c) Compare the results you have obtained for (a) and (b). What theorem of vector analysis is relevant to the conclusion you draw from the comparison?

3. The Laplace transform of a function $f(t)$ is defined by [20 points]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s),$$

where s is chosen so as to make the improper integral above converge.

(a) Given that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$ and $\mathcal{L}\{\cos at\} = s/(s^2 + a^2)$ with $s > 0$, what is the Laplace transform of the convolution integral $f(t) = \int_0^t \sin(t - \tau) \cos 2\tau \, d\tau$.

(b) Assuming that the Laplace transform can be computed term by term, evaluate the Laplace tranform of $J_0(t) = \sum_0^\infty \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$, the Bessel function of the first kind of order zero. You are supposed to sum the resulting series of the transform done term by term.

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4. Consider the differential equation [20 points]

$$L[y] \equiv y''' - 4y' = t + 3\cos t + e^{-2t},$$

where y' and y''' represent respectively the first and the third derivatives of y with respect to the independent variable t.

(a) Find the general solution of the corresponding homogeneous equation $L[y] = 0$.

(b) Find a particular solution of the inhomogeneous equation given above.

(c) What is the general solution of the inhomogeneous equation given above?

5. Consider the Legendre equation [20 points]

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0\,.$$

(a) Determine the radii of convergence of the series solutions about $x = 0$ for the Legendre equation.

(b) Show that the series solutions of the Legendre equation become polynomials of degree ℓ if α is equal to the integer ℓ.

The Legendre polynomial $P_\ell(x)$ of order ℓ (with ℓ being an integer) is defined as the polynomial solution of the Legendre equation above with $\alpha = \ell$ that also satisfies the condition $P_\ell(1) = 1$.

Furthermore, the Legendre polynomials satisfy the following orthogonality property:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}.$$

(c) A polynomial of degree n, say $f(x)$, defined on the interval $-1 \le x \le 1$ can be expressed as

$$f(x) = \sum_{\ell=0}^{n} c_\ell P_\ell(x)\,.$$

Find the expansion coefficients c_ℓ for $f(x) = 3x^2 + x - 1$.