1. (35%) Consider a mass m_1 suspended vertically from a rigid support by a weightless spring with a second mass m_2 suspended from the first mass by means of a second weightless spring as shown in the figure 1. Let's treat the two masses as point masses, and assume that the two springs obey Hooke's law with spring constants k_1 and k_2, respectively.

(i) Derive the equations governing the displacements for the two masses, $x_1(t)$ and $x_2(t)$, using Newton's second law with negligible air resistance. Here t denotes the time. The result should be

$$m_1\frac{d^2x_1}{dt^2}=-k_1x_1(t)+k_2[x_2(t)-x_1(t)],$$
$$m_2\frac{d^2x_2}{dt^2}=-k_2[x_2(t)-x_1(t)].$$
(10%)

(ii) Solve the equations for $x_1(t)$ and $x_2(t)$ in (i) with $m_1=m_2=1,\ k_1=5,\ k_2=6$, subject to $x_1=2,\ \frac{dx_1}{dt}=5,\ x_2=-10,\ \frac{dx_2}{dt}=1$ at $t=0$. Calculate the speeds of the two masses. (15%)

(iii) How do we modify the equations in (i) if an external force F(t) is applied to mass m_2 along the direction of x_2 and the drag associated with the air resistance is included? Assume that the drag on the mass is proportional to its velocity but in opposite direction. (10%).

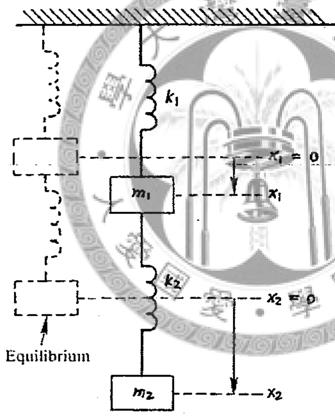

Figure 1

2. (30%) Consider a full-rank ($m\times n$) real-valued matrix $\mathbf{A}$ ($m\ge n$). Define $\text{Range}(\mathbf{A})=\{\mathbf{y}\in R^m:\exists\mathbf{x}\in R^n \ni \mathbf{y}=\mathbf{Ax}\}$ and $\text{Ker}(\mathbf{A}^T)=\{\mathbf{y}\in R^m:\mathbf{A}^T\mathbf{y}=\mathbf{0}\}$, where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$, and R^m is the m-dimensional vector space of real numbers. Show that

(i) Any vector in R^m can be expressed as a sum of a vector in $\text{Range}(\mathbf{A})$ and a vector in $\text{Ker}(\mathbf{A})$. (10%)

(ii) $\text{Range}(\mathbf{A})\cap\text{Ker}(\mathbf{A})=\{\mathbf{0}\}$. (10%)

(iii) Either the solution of $\mathbf{Ax}=\mathbf{y}$ exists for all $\mathbf{y}\in R^m$, or the problem $\mathbf{A}^T\mathbf{y}=\mathbf{0}$ has non-zero solutions. (10%)

3. (35%)

(i) (a) Determine the regions where the following partial differential equation is of elliptic, parabolic, or hyperbolic type. (6%)

$$u_{xx}+yu_{yy}=0$$

(b) Obtain its characteristics and its canonical form for each region in (a). (9%)

(ii) Solve the partial differential equation

$$u_{tt}-u_{xx}=0,\ (0<x<\infty),$$

subject to the conditions

$$u(0,t)=H(t)\exp(-t),\ u(x,0)=u_t(x,0)=0,$$

where $H(t)$ is the Heaviside unit step function. (20%)

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