1. Consider the real linear vector space, V, which consists of all real-coefficient polynomials in t of degree ≤ 2. Answer the following questions.

   (1) (3%) What is the dimension of V?

   (2) (3%) Find the components of the vector $k(t) = 1 - 2t + t^2$ with respect to the f-basis $\left\{f_1(t) = t^2,\ f_2(t) = 2 + t,\ f_3(t) = t - 2t^2\right\}$ for V. Denote it as $(k)_f$.

   (3) (4%) Find the transformation matrix (P) from the f-basis to the standard basis $\left\{e_1(t) = 1,\ e_2(t) = t,\ e_3(t) = t^2\right\}$ for V, that is, $(k)_e = P(k)_f$, where $(k)_e$ is the coordinates of the vector $k(t)$ with respect to the standard basis.

2. (15%) Consider the initial-value problem:

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{d^2y}{dt^2} = x - y + \dfrac{dy}{dt} \end{cases} \quad \text{with} \quad x(0) = 0,\ y(0) = 1,\ \frac{dy}{dt}(0) = 0.$$

   Solve the problem in use of the method of Laplace transform.

3. (15%) Find the general solution of the following ordinary differential equation

$$x\frac{d^2y}{dx^2} + \left(2x^2 - 3\right)\frac{dy}{dx} + \left(x^3 - 2x + 3x^{-1}\right)y = x^6 \quad \text{for } x>0$$

   by performing the change of variables $y(x) = xU(t)$ and $x = \sqrt{t}$.

4. For each of the following Fourier series expansion:

$$f_i(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \quad \text{for } -\pi < x < \pi,$$

$$f_{ii}(x) = 1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[(2n-1)\frac{x}{2}\right] \quad \text{for } 0 < x < 2\pi;$$

$$f_{iii}(x) = x = k - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[(2n-1)\frac{x}{2}\right] \quad \text{for } 0 \leq x \leq 2\pi$$

   (1) (3%) What is the numerical value of $\int_0^{2\pi} f_{iii}(x)\cos(65x/2)dx$?

   (2) (3%) What is the numerical value of k in $f_{iii}(x)$?

   (3) (3%) What are the numerical values of each series at $x = \pi/3$, π, and 12.5π? (9 answers required)

   (4) (3%) Find the Fourier series for $|x|$, $-2\pi < x < 2\pi$.

   (5) (3%) Does $\int x dx = \dfrac{x^2}{2} = 2\sum_{n=1}^{\infty}(-1)^n n^2(\cos nx - 1)$; $-\pi < x < \pi$?

   Does $\dfrac{dx}{dx} = 1 = 2\sum_{n=1}^{\infty}\cos nx$; $-\pi < x < \pi$?

   Give reasons why you answered "yes" or "no" to these questions.

5. Consider the following 1-dimensional heat equation:

$$\frac{\partial^2 u}{\partial x^2} = u + \frac{\partial u}{\partial t} \quad \text{for } 0 < x < 2,\ t > 0$$

   with initial condition $u(x,0) = \sin\dfrac{\pi x}{4}$, $0 \leq x \leq 2$.

   (1) (4%) For $t \geq 0$, the wire temperature is kept zero at $x = 0$ and the wire is insulated at $x = 2$. Write down the mathematical form of these boundary conditions.

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(2) (4%) Following (1), use separation of variables on this heat equation to obtain two ordinary differential equations for $X(x)$ and $T(t)$, and the boundary conditions for $X(x)$.

(3) (4%) Following (2), find all non-trivial solutions of $X(x)$.

(4) (3%) Which, if any, of the equations given below is a solution to the given heat equation with the given boundary conditions in part (1). (Justification of your answer is required to get credit.)

(i) $u(x,t)=e^{(-1-\frac{\pi^2}{16})t}\sin\frac{\pi x}{4}$ (ii) $u(x,t)=e^{\frac{-\pi^2 t}{16}}\sin\frac{\pi x}{4}$

(iii) $u(x,t)=e^{\frac{-\pi^2 t}{16}}\cos\pi x$

(iv) $u(x,t)=\sum_{n=1}^{\infty} b_n e^{\frac{-n^2\pi^2 t}{16}}\sin\frac{n\pi x}{4}$, $b_n=\int_0^2 \sin(\frac{\pi x}{4})\sin(\frac{n\pi x}{2})dx$

6. Write down the answers to the following questions. (Derivations are not required.)

(1) (3%) Evaluate the line integral $\oint_C \underline{\mathbf{F}}\cdot\underline{\mathbf{n}}\,d\ell$ of a 2-D vector function $\underline{\mathbf{F}}=(\frac{x}{x^2+y^2})\,\underline{\mathbf{i}}+(\frac{y}{x^2+y^2})\,\underline{\mathbf{j}}$ over a closed path C defined by an ellipse $9x^2+4y^2=1$. ($\underline{\mathbf{n}}$ denotes the unit normal vector pointing outwardly along the ellipse.)

(2) (3%) Let $\phi(x,y,z)=xyz$ be a scalar function. Evaluate the surface integral $\oint\!\!\oint_S (\nabla\phi)\cdot\underline{\mathbf{n}}\,dS$ over the bounding surface S of a cube defined by $-1\le x\le 1$, $-1\le y\le 1$, $-1\le z\le 1$. ($\underline{\mathbf{n}}$ denotes the unit normal vector pointing outwardly along the surface of the cube.)

(3) (3%) Let $\underline{\mathbf{F}}=(2x^2-y)\,\underline{\mathbf{i}}+(\cos y-ye^{-y}+4x)\,\underline{\mathbf{j}}$ be a 2-D vector function. Evaluate the line integral $\oint_C \underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ along a closed path C defined by a unit circle centered at the origin.

(4) (3%) Let $\phi(x,y,z)=x^2y-xe^z$. Find the rate of change of $\phi(x,y,z)$ at point (1, 0, -1) along the direction $\underline{\mathbf{u}}=\underline{\mathbf{i}}-2\underline{\mathbf{j}}+\underline{\mathbf{k}}$.

(5) (3%) Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be two continuous and differentiable scalar functions, then $\nabla\cdot(\nabla\phi\times\nabla\psi)=0$. True or False ?

7. Let $z=x+iy$ denote the complex variable, $\bar{z}=x-iy$ be the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. (Derivations are not required.)

(1) (3%) $f(z)=\bar{z}/z$ is an analytic function on the whole z-plane excluding the origin. True or False ?

(2) (3%) Find the residue of the complex function $f(z)=z(z+i)e^{1/z^2}$ at $z=0$.

(3) (3%) Let the Laurent series expansion of $f(z)=(z+3i)/[z(z^2+9)]$ about $z=3i$ be denoted by $\sum_{n=-\infty}^{n=+\infty} c_n(z-3i)^n$ which is a convergent series within the annulus $0<|z-3i|<3$. Find the sum of the coefficients c_n of all negative-power terms; i.e., evaluate $\sum_{n=-\infty}^{n=-1} c_n=?$

(4) (3%) Evaluate the complex integral $\oint_C[(\sin z)/(z-i)^2]\,dz$ over C: $|z-i|=2$.

(5) (3%) Evaluate the real integral $\int_0^{2\pi} e^{(\cos\theta)}\cos(\sin\theta)\,d\theta$.

<hint>: Consider first the complex integral $\oint_C (e^z/z)\,dz$ along a unit circle C centered at origin.

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