Note: You can use either Chinese or English to answer the questions below. The total number of points is 100. The points for each problem is indicated in the parenthesis. You need to show all the work to receive the points.

1. Let U and V be independent random variables such that $U \sim \chi^2_n$ and $V \sim \chi^2_m$.
   (a) (10 points) Find the distribution of $(U/n)/(V/m)$.
   (b) (10 points) Determine $\mathrm{E}(U \exp(V/4))$.

   Hint: Chi Squared Distribution, χ^2_k

$$\text{pdf: } f(x|k) = \frac{1}{\Gamma(k/2)2^{k/2}} x^{(k/2)-1} e^{-x/2}, \quad x > 0; \quad \text{mean} = k; \quad \text{variance} = 2k$$

2. Let $X_1, \ldots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution, where μ and σ^2 are unknown parameters. Define $\bar{X}$ and T by

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad T = \frac{1}{n}\sum_{i=1}^{n}(X_i/\sigma)^2.$$

   (a) (10 points) Is $\bar{X}$ a statistic? Is T a statistic? Explain.
   (b) (10 points) Determine $\mathrm{E}(\bar{X})$ and $\mathrm{E}(T)$.
   (c) (10 points) Determine $\mathrm{Cov}(\bar{X}, T)$.

   Hint: Normal Distribution, $N(\mu, \sigma^2)$

$$\text{pdf: } f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]; \quad \text{mgf} = \exp[\mu t + \sigma^2 t^2/2]$$

3. Assume $X_1, \ldots, X_n$ are iid with density

$$f(x|\alpha) = \frac{2x}{\alpha} e^{-x^2/\alpha}, \quad x > 0, \quad \alpha > 0.$$

   (a) (10 points) Identify a minimal sufficient statistic. Is it complete? Justify your answer.
   (b) (10 points) Find a maximum likelihood estimator of α. Is it unique?

4. Suppose $b_1, \ldots, b_n$ are known constants and $W_i \sim \text{normal}(b_i\mu, 1)$, $i = 1, \ldots, n$ are independent random variables.
   (a) (10 points) Identify the maximum likelihood estimator for μ, and its distribution.
   (b) (10 points) Find the uniformly most powerful test of $H_0 : \mu \le \mu_0$ vs. $H_1 : \mu > \mu_0$.
   (c) (10 points) Each of the following are unbiased estimators of μ. Which should you use and why?

$$\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{W_i}{b_i}, \quad \hat{\theta}_2 = \frac{\sum_{i=1}^{n} W_i}{\sum_{i=1}^{n} b_i}, \quad \hat{\theta}_3 = \frac{\sum_{i=1}^{n} b_i W_i}{\sum_{i=1}^{n} b_i^2}$$