1 (10分) (a) (5分) Write down the Taylor expansions of e^{-x}. What is the e^{-1} scale for the functions of e^{-k^2t} and $e^{-x^2/n}$?

(b) (5分) Draw a graph for $f(t) = t^{100}e^{-t}$ $(t \geq 0)$ and explain the function behavior. What is the t when the function has a maximum?

2 (10分) Consider the dynamic system:

$$\frac{d^2x_1}{dt^2} = -(x_1 - x_2),$$
$$\frac{d^2x_2}{dt^2} = -(x_2 - x_1).$$

Write the above equation in the matrix form, and discuss the fundamental types of motion and the corresponding periods of the system? (hint: find the eigenvalues and eigenvectors of the matrix.)

3(10分) Find the Fourier transform for the function $f(x) = e^{-a|x|}$, where a is real and $a > 0$, $-\infty < x < \infty$.

4 (10分) Solve the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

in $[0, \infty] \times [0, \pi]$ domain with the boundary conditions $\psi(x, 0) = \psi(x, \pi) = 0$, $\psi(0, y) = f_0(y)$, and $\psi(\infty, y) = 0$. Discuss the smoothing effect of the Laplace equation in the x direction.

5 (10分) Solve the following eigenvalue and eigenvector problem.

$$\frac{d^2u}{dx^2} = -n^2u$$

with $u(0) = u(\pi)$.

6 (15分) Solve the following differential equations:

(a) Gompertz equation

$$\frac{dT}{dt} = \alpha \ln(\frac{\mu}{T})T,$$

with $T(0) = T_0$, α and μ are constants.

(b)

$$\frac{dy}{dt} + \lambda y = e^{it},$$

with $y(0) = y_0$.

(c)

$$\frac{d^2u}{dx^2} = \begin{cases} 2, & \text{if} \quad -1 \leq x < 0; \\ -2, & \text{if} \quad 1 \geq x > 0. \end{cases}$$

with $u(-1) = u(1) = 0$.

7 (15分) Express the following vector operations in the Cartesian components, and also state whether the yield of vector operation is a scalar or a vector. (Boldface **V** is a vector.)

(a) $\nabla \cdot \mathbf{V}$, (b) $\nabla \times \mathbf{V}$, (c) $\nabla\phi$, (d) $\nabla^2\phi$, (e) $\mathbf{V} \cdot \nabla\phi$.

8 (10分) State the Stokes' theorem and the Gauss' theorem in equations and discuss the meanings.

9 (10分) Prove the Leibniz integration equation

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt + f(x, b(x))\frac{db(x)}{dx} - f(x, a(x))\frac{da(x)}{dx}.$$

試題隨卷繳回