Problem 1 (25%)

It is known that $A=\begin{bmatrix}2 & 1 & -1\\ 3 & 2 & -3\\ 3 & 1 & -2\end{bmatrix}$. If $B = A^6-2A^5-3A^4+9A^3-4A^2-6A+8I$, find **B** and e^{B}, both solutions should be expressed in terms of a 3 x 3 matrix.

Problem 2 (25%)

The Fourier series representation of $g(x)=|x|$, for $-L\le x\le L$, has been found to be

$$g(x)=\frac{L}{2}-\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos[\frac{(2n-1)\pi x}{L}]}{(2n-1)^2}.$$

Given $f(x)=\begin{cases}1 & for\ x<0\\ -1 & for\ 0<x\end{cases}$ and $h(x)=\int_{-2\pi}^{x} f(t)\,dt$.

Find the Fourier series representation of $h(x)=\int_{-2\pi}^{x} f(t)\,dt$, for $-2\pi\le x\le 2\pi$.

Problem 3 (25%)

(a) Solve the following ordinary differential equation

$\frac{d^2\phi(x)}{dx^2}=1+x \quad -1\le x\le 1$, which is subjected to the boundary conditions:

$\phi(x=-1)=0$ and $\phi(x=1)=0$.

And identify the homogeneous, particular solutions and resonant modes.

(b) Express the solution of $\phi(x)$ from *Part(a)* in terms of the Chebyshev polynomials $T_n(x)$.

Hint: $T_0(x)=1$; $T_1(x)=x$; $T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0$; $(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0$;

$$\int_{-1}^{1}\frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}dx=0,\ m\ne n;\ \|T_0(x)\|^2=\pi;\ \|T_n(x)\|^2=\pi/2;\ n=1,2,3,\dots.$$

Problem 4 (25%)

(a) Find the solution of the following partial differential equation (PDE):

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\ \ r_1=3\le r\le r_2=5$$

Where $u(r,\theta)$ is periodic in θ with period 2π and subject to the Dirichlet boundary conditions: $u(r_1=3,\theta)=F(\theta)=2+\cos\theta$; $u(r_2=5,\theta)=G(\theta)=1$

(b) Explain the applications of this PDE.

Hint: You may assume $u(r,\theta)=R(r)\,\Theta(\theta)$.

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