※ 注意：請於試卷上依序作答，並應註明作答之大題及其題號。

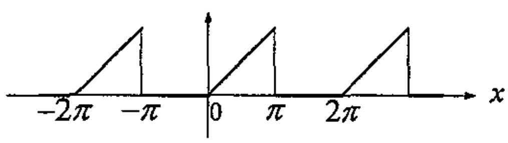
1. (10%) The initial conditions $y(0)=y_0$, $y'(0)=y_1$, apply to the following differential equation:
$x^2y''-4xy'+4y=0$.
For what values of y_0 and y_1 does the initial-value problem have a solution ?

2. Consider the matrix $\mathbf{A}=\begin{bmatrix}0&2&1&0\\0&0&2&1\\0&0&0&2\\0&0&0&0\end{bmatrix}$.
   (1) (6%) Determine the rank of **A**.
   (2) (6%) Let **B** be a 4×3 matrix satisfying $\mathbf{AB}=\mathbf{0}$. Find the maximum possible value of the rank of **B**.
   (3) (6%) Find the eigenvalues and eigenvectors of **A**.
   (4) (6%) Is **A** diagonalizable ?
   (5) (6%) Let $\mathbf{b}=[b_1,b_2,b_3,b_4]^T$. Under what conditions on **b** (if any) does $\mathbf{Ax}=\mathbf{b}$ have a solution ?

3. Let $u(x,t)$ denote the displacement of a finite string over $0<x<\pi$ with fixed ends, $u(0,t)=u(\pi,t)=0$. The string starts to vibrate from its initial states, $u(x,0)=0$ and $u_t(x,0)=(\partial u/\partial t)_{t=0}=x$, after an external force, $F(x)=x(x-\pi)$, is applied onto it. The subsequent string displacement can be described by a 1-D inhomogeneous wave equation:
$$\frac{\partial^2u}{\partial t^2}=9\frac{\partial^2u}{\partial x^2}+F(x)\quad,\text{ for }0<t<\infty.$$
   (1) (4%) Write $u(x,t)=X(x)T(t)$. Use the homogeneous problem (set $F(x)=0$ with the same boundary conditions) and the Sturm-Liouville theorem to determine the eigenfunction and the corresponding eigenvalue, $\phi_n(x)$ and λ_n, of the problem.
   (2) (4%) Find the eigenfunction expansion of the external force: determine the coefficients f_n's in $F(x)=\sum_{n=1}^{\infty}f_n\phi_n(x)$.

   Use (1) and (2) to find a solution for the original wave equation in the form of $u(x,t)=\sum_{n=1}^{\infty}T_n(t)\phi_n(x)$:
   (3) (4%) Determine a 2nd order ODE (ordinary differential equation) that governs $T_n(t)$.
   (4) (5%) Apply the initial conditions to solve the ODE obtained in (3). Then, complete the solution to the 1-D inhomogeneous wave equation.

4. A periodic function $f(x)$ is sketched below.



   (1) (2%) Write a mathematical description for the function.
   (2) (3%) Determine the Fourier series representation, $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(\lambda_nx)+b_n\sin(\lambda_nx)$, of the function: explicitly write out a_0, a_n, b_n, λ_n.
   (3) (3%) Determine the fundamental period, ω_0, of the Fourier series. Sketch the amplitude spectrum of the Fourier coefficients over the frequency range of $[0,4\omega_0]$.

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(4) (2%) What is Gibbs phenomenon ?

(5) (3%) Is Gibbs phenomenon present in the Fourier series representation for $f(x)$ found in (2)? If yes, indicate the location where Gibbs phenomenon will be most pronounced over the range $-5 < x < 5$. Otherwise, explain why the current Fourier series representation is free of Gibbs phenomenon.

5. Write down the answers to the following questions. (Derivations are not required.)

(1) (3%) Evaluate the distance from the point (1, 3, 0) to the plane: $x - 3y + \sqrt{6}z = 3$.

(2) (3%) Let $\phi(x, y, z) = xyz + x^2 - 2y^2$ be a scalar function. Evaluate the flux of $\nabla\phi$ out of the surface of the sphere: $x^2 + y^2 + z^2 = 4$.

(3) (3%) Let $\underline{\mathbf{F}} = r\,\underline{\mathbf{e}}_r + r\cos\theta\,\underline{\mathbf{e}}_\theta + z\,\underline{\mathbf{e}}_z$ be a vector function written in the cylindrical coordinates (r, θ, z). Evaluate $\nabla \cdot \underline{\mathbf{F}}$.

(4) (3%) Find the streamlines of the 2-D vector field $\underline{\mathbf{F}} = \sin(2y)\,\underline{\mathbf{i}} + \cos(x)\,\underline{\mathbf{j}}$.

(5) (3%) Let $\phi(x, y)$ and $\psi(x, y)$ be two continuous and differentiable scalar functions on a simple closed curve C and throughout the interior D of C, then $\oint_C -\phi(\partial\psi/\partial y)\,dx + \phi(\partial\psi/\partial x)\,dy = \iint_D \phi\nabla^2\psi\,dA + \iint_D [\quad]\,dA$. Fill in the blank bracket [ ] in the above identity with proper expression.

6. Let $z = x + iy$ denote the complex variable, $\bar{z} = x - iy$ the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. (Derivations are not required.)

(1) (3%) Find the real part of $(1-i)^{(1+i)}$ if the argument θ is restricted in $0 \le \theta < 2\pi$.

(2) (3%) Find the residue of the complex function $f(z) = z(z-i)\cos(1/z^2)$ at $z = 0$.

(3) (3%) Let the Laurent series expansion of $f(z) = (z+i)/(z^2+4)$ about $z = -2i$ be denoted by $\sum_{n=-\infty}^{n=+\infty} c_n (z+2i)^n$ which is a convergent series within the annulus $0 < |z + 2i| < 4$. Find the sum of the coefficients c_n of all negative-power terms; i.e., evaluate $\sum_{n=-\infty}^{n=-1} c_n = ?$

(4) (3%) Evaluate the complex integral $\oint_C [\bar{z}/(z+2i)^2]\,dz$ over $C: |z| = 1$.

(5) (3%) Evaluate the real integral $\int_{\theta=0}^{\theta=2\pi} \frac{d\theta}{\cos\theta + \sin\theta}$.