Note: You can use either Chinese or English to answer the questions below. The total number of points is 100. The points for each problem is indicated in the parenthesis. You need to show all the work to receive the points.

1. The random variable X has the $beta(a,b)$ distribution, and the conditional distribution of Y given that $X=x$ is $Bernoulli(x)$.

   (a) (10 points) Find Var(Y).

   (b) (10 points) Find Var[E($Y|X$)].

   Hint: Beta distribution, $beta(a,b)$

   $$\text{pdf: } f_X(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}, \quad 0\le x\le 1, \qquad \text{where } \Gamma(r)=r!;$$

   Bernoulli distribution, $Bernoulli(p)$

   $$\text{pmf: } f_X(x)=p^x(1-p)^{1-x}, \quad x=0,1, \quad 0\le p\le 1;$$

2. A box contains eight balls, five red and three white. Suppose three of the eight balls are randomly selected and put in a bag.

   (a) (10 points) What is the probability that the bag contains 3 red balls?

   (b) (10 points) If a ball is randomly selected from the five remaining balls in the box, what is the probability it is white?

   (c) (10 points) Suppose it is known that at least one of the three balls that were put in the bag is white. Now what is the probability that the ball randomly selected from the remaining five is white?

3. Let $X_1,\ldots,X_n$ be a random sample from the distribution with pdf

   $$f(x|\theta)=\theta x^{\theta-1}, \quad 0<x<1, \quad \theta>0$$

   (a) (10 points) Obtain a method of moments estimator of θ.

   (b) (10 points) Obtain the maximum likelihood estimator of θ.

   (c) (10 points) Obtain the Cramef-Rao lower bound for the variance of unbiased estimators of θ.

4. Let X be a single observation from a distribution with density

   $$f(x|\theta)=\begin{cases} 0.5, & -1<x<1, \quad \text{if } \theta=0 \\ \frac{3}{4}(1-x^2), & -1<x<1, \quad \text{if } \theta=1 \end{cases}$$

   (a) (10 points) Consider testing $H_0:\theta=0$ versus $H_1:\theta=1$. Find the level of significance and power of the test that has rejection region $R=\{x:x>0.8\}$.

   (b) (10 points) Prove or disprove the test in part (a) is the most powerful test of its size for testing H_0 versus H_1.