※ 注意：請用 2B 鉛筆作答於答案卡，並先詳閱答案卡上之「畫記說明」。

選擇題 (單選; 每題答對得 2 分; 答錯或未答得 0 分)

1. The n-th Catalan number is given directly in terms of binomial coefficients by:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} \qquad \text{for } n \geq 0.$$

It can also be given by the following alternative expression:

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} \qquad \text{for } n \geq 1.$$

The Catalan numbers satisfy the recurrence relation:

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \qquad \text{for } n \geq 0.$$

They also satisfy:

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n.$$

The first Catalan numbers for $n = 0, 1, 2, 3, \ldots$ are: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$ Which of the following is **not** an interpretation of the n-th Catalan number for $n \geq 1$?

(A) The number of lattice paths of length $n-1$ from $(0,0)$ to the x-axis with steps $(\pm 1, 0)$ and $(0, \pm 1)$, never going below the x-axis;

(B) The number of plane trees with n vertices whose leaves at height one are colored red or blue;

(C) The number of vertices of height $n-1$ of the tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees $2, 3, \ldots, k+1$;

(D) The number of movements required to solve the three-peg, n-disk Tower of Hanoi problem;

(E) The number of expressions containing n pairs of parentheses which are correctly matched (for example, (()()) is a correctly-matched expression, while ())(() is not).

2. Let X be a set of four elements, and let $\mathcal{B}_X$ be the set of all bijective functions from X to itself. Clearly, $\mathcal{B}_X \neq \emptyset$, since the identity map ι_X from X to itself is in $\mathcal{B}_X$. Let N be the set of positive integers n such that $\forall f \in \mathcal{B}_X, f^n = \iota_X$. Let $\hat{n}$ be the smallest element of N. *Which of the following statements is* ***true***?

(A) $\hat{n} = 1$;

(B) $\hat{n} = 4$;

(C) $\hat{n} = 6$;

(D) $\hat{n} = 12$;

(E) $\hat{n}$ does not exist because $N = \emptyset$.

3. Let $b > 1$. Then $\log_b((n^2)!) = ?$

(A) $\Theta(\log_b(n!))$;

(B) $\Theta(\log_b(2n!))$;

(C) $\Theta(n \log_b(n))$;

(D) $\Theta(n^2 \log_b(n))$;

(E) $\Theta(n \log_b(n^2))$.

4. $\sum_{i=1}^{n} i^{-1/2} = ?$

(A) $\Theta((\ln(n))^{1/2})$;

(B) $\Theta(\ln(n))$;

(C) $\Theta(n^{1/2})$;

(D) $\Theta(n^{3/2})$;

(E) $\Theta(n^2)$.

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5. Let G be a *simple* graph. Then which of the following statements could possibly be **true**?

(A) G has 3 components, 20 vertices, and 16 edges;
(B) G has 6 vertices, 11 edges, and more than one component;
(C) G is connected and has 10 edges, 5 vertices, and fewer than 6 cycles;
(D) G has 7 vertices, 10 edges, and more than two components;
(E) G has 8 vertices, 8 edges, and no cycles.

6. A *clique* in a simple (undirected) graph is a complete subgraph that is not contained in any larger complete subgraph. Let G be the graph represented by the adjacency matrix below. How many cliques does G have?

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

(A) 5;
(B) 6;
(C) 7;
(D) 8;
(E) 9.

7. Let $S = \{0,1,2,3,4,5,6,7,8,9\}$. What is the smallest integer K such that any subset of S of size K contains two disjoint subsets of size two, $\{x_1, x_2\}$ and $\{y_1, y_2\}$, such that $x_1 + x_2 = y_1 + y_2 = 9$?

(A) 5;
(B) 6;
(C) 7;
(D) 8;
(E) 9.

8. Given that $k > 1$, which of the following sum or product representations is **false**?

(A) $(2^2+1)(3^2+1)\cdots(k^2+1) = \prod_{j=2}^{k}[(j+1)^2 - 2j]$;
(B) $(1^3-1)+(2^3-2)+\cdots+(k^3-k) = \sum_{j=1}^{k-1}[(k-j)^3-(k-j)]$;
(C) $(1-r)(1-r^2)(1-r^3)\cdots(1-r^k) = \prod_{j=0}^{k-1}(1-r^{k-j})$;
(D) $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k-1}{k!} = \sum_{j=2}^{k}\frac{j-1}{j!}$;
(E) $n+(n-1)+(n-2)+\cdots+(n-k) = \sum_{j=1}^{k+1}(n-j+1)$.

9. Which of the following statements is **true**?

(A) A number is rational if and only if its square is rational;
(B) An integer n is odd if and only if n^2+2n is odd;
(C) A number is irrational if and only if its square is irrational;
(D) A number n is odd if and only if $n(n+1)$ is even;
(E) At least one of two numbers x and y is irrational if and only if the product xy is irrational.

10. For all $N \geq 0$, if $N = k(k+1)(k+2)$ is the product of three consecutive non-negative integers, then for some integer $s > k$, N is divisible by a number of the form:

(A) s^2-1;
(B) s^2-2;
(C) s^2;
(D) s^2+1;
(E) s^2+2.

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11. The number of primes of the form $|n^2 - 6n + 5|$, where n is an integer, is:

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

12. The Euclidean Algorithm is used to produce a sequence $X_1 > X_2 > X_3 > X_4 > X_5 = 0$ of positive integers, where $X_t = q_{t+1}X_{t+1} + X_{t+2}, t = 1, 2, 3$. The quotients are $q_2 = 3$, $q_3 = 2$, and $q_4 = 2$. Which of the following is correct?

(A) $\gcd(X_1, X_2) = -2X_1 + 6X_2$;
(B) $\gcd(X_1, X_2) = -2X_1 - 6X_2$;
(C) $\gcd(X_1, X_2) = -2X_1 - 7X_2$;
(D) $\gcd(X_1, X_2) = 2X_1 + 7X_2$;
(E) $\gcd(X_1, X_2) = -2X_1 + 7X_2$.

13. Which of the following statements is the *contrapositive* of the statement, "You win the game if you know the rules but are not overconfident."

(A) If you lose the game then you don't know the rules or you are overconfident;
(B) A sufficient condition that you win the game is that you know the rules or you are not overconfident;
(C) If you don't know the rules or are overconfident, then you lose the game;
(D) If you know the rules and are overconfident, then you win the game;
(E) A necessary condition that you know the rules or you are not overconfident is that you win the game.

14. Consider the statement, "Given that people who are in need of refuge and consolation are apt to do odd things, it is clear that people who are apt to do odd things are in need of refuge and consolation." This statement, of the form $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$, is logically equivalent to:

(A) People who are in need of refuge and consolation are not apt to do odd things;
(B) People are apt to do odd things if and only if they are in need of refuge and consolation;
(C) People who are apt to do odd things are in need of refuge and consolation;
(D) People who are in need of refuge and consolation are apt to do odd things;
(E) People who aren't apt to do odd things are not in need of refuge and consolation.

15. The truth table for a Boolean expression is specified by the correspondence $(P, Q, R) \mapsto S$, where:

$$
\begin{aligned}
(0,0,0) &\mapsto 0,\\
(0,0,1) &\mapsto 1,\\
(0,1,0) &\mapsto 0,\\
(0,1,1) &\mapsto 1,\\
(1,0,0) &\mapsto 0,\\
(1,0,1) &\mapsto 0,\\
(1,1,0) &\mapsto 0,\\
(1,1,1) &\mapsto 1.
\end{aligned}
$$

A Boolean expression having this truth table is:

(A) $[(\sim P \wedge \sim Q) \vee Q] \vee R$;
(B) $[(\sim P \wedge \sim Q) \wedge Q] \wedge R$;
(C) $[(\sim P \wedge \sim Q) \vee \sim Q] \wedge R$;
(D) $[(\sim P \wedge \sim Q) \vee Q] \wedge R$;
(E) $[(\sim P \vee \sim Q) \wedge Q] \wedge R$.

16. Which of the following statements is **false**?

(A) $2 \in A \cup B$ implies that if $2 \notin A$ then $2 \in B$;
(B) $\{2,3\} \subseteq A$ implies that $2 \in A$ and $3 \in A$;
(C) $A \cap B \supseteq \{2,3\}$ implies that $\{2,3\} \subseteq A$ and $\{2,3\} \subseteq B$;
(D) $A - B \supseteq \{3\}$ and $\{2\} \subseteq B$ implies that $\{2,3\} \subseteq A \cup B$;
(E) $\{2\} \in A$ and $\{3\} \in A$ implies that $\{2,3\} \subseteq A$.

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17. Which of the following statements is **true**?

(A) For all sets A, B, and C, $A-(B-C)=(A-B)-C$;
(B) For all sets A, B, and C, $(A-B)\cap(C-B)=(A\cap C)-B$;
(C) For all sets A, B, and C, $(A-B)\cap(C-B)=A-(B\cup C)$;
(D) For all sets A, B, and C, if $A\cap C=B\cap C$ then $A=B$;
(E) For all sets A, B, and C, if $A\cup C=B\cup C$ then $A=B$;

18. Which of the following statements is **false**?

(A) $C-(B\cup A)=(C-B)-A$;
(B) $A-(C\cup B)=(A-B)-C$;
(C) $B-(A\cup C)=(B-C)-A$;
(D) $A-(B\cup C)=(B-C)-A$;
(E) $A-(B\cup C)=(A-C)-B$.

19. The power set $\mathcal{P}((A\times B)\cup(B\times A))$ has the same number of elements as the power set $\mathcal{P}((A\times B)\cup(A\times B))$ if and only if:

(A) $A=B$;
(B) $A=\emptyset$ or $B=\emptyset$;
(C) $B=\emptyset$ or $A=B$;
(D) $A=\emptyset$ or $B=\emptyset$ or $A=B$;
(E) $A=\emptyset$ or $B=\emptyset$ or $A\cap B=\emptyset$.

20. Let $f:X\to Y$. Consider the statement, "For all subsets C and D of Y, $f^{-1}(C\cap D^c)=f^{-1}(C)\cap[f^{-1}(D)]^c$." This statement is:

(A) True and equivalent to: "For all subsets C and D of Y, $f^{-1}(C-D)=f^{-1}(C)-f^{-1}(D)$;"
(B) False and equivalent to: "For all subsets C and D of Y, $f^{-1}(C-D)=f^{-1}(C)-f^{-1}(D)$;"
(C) True and equivalent to: "For all subsets C and D of Y, $f^{-1}(C-D)=f^{-1}(C)-[f^{-1}(D)]^c$;"
(D) False and equivalent to: "For all subsets C and D of Y, $f^{-1}(C-D)=f^{-1}(C)-[f^{-1}(D)]^c$;"
(E) True and equivalent to: "For all subsets C and D of Y, $f^{-1}(C-D)=[f^{-1}(C)]^c-f^{-1}(D)$."

21. Let $\mathbb{Z}$ be the set of all integers. Define $f(n)=\frac{n}{2}+\frac{1-(-1)^n}{4}$ for all $n\in\mathbb{Z}$. Which of the following statements is **true**?

(A) f is not a function from $\mathbb{Z}$ to itself because $\frac{n}{2}\notin Z$;
(B) f is a function and is onto and one-to-one;
(C) f is a function and is not onto but is one-to-one;
(D) f is a function and is not onto and not one-to-one;
(E) f is a function and is onto but not one-to-one.

22. The number of partitions of $\{1,2,3,4,5\}$ into three blocks is $\binom{5}{3}=25$. The total number of functions $f:\{1,2,3,4,5\}\to\{1,2,3,4\}$ with $|f(\{1,2,3,4,5\})|=3$ is:

(A) 4×6;
(B) 4×25;
(C) 25×6;
(D) $4\times 25\times 6$;
(E) $3\times 25\times 6$.

23. Let $f:X\to Y$ and $g:Y\to Z$. Let $h=g\circ f:X\to Z$. Suppose g is one-to-one and onto. Which of the following is **false**?

(A) If f is one-to-one then h is one-to-one and onto;
(B) If f is not onto then h is not onto;
(C) If f is not one-to-one then h is not one-to-one;
(D) If f is one-to-one then h is one-to-one;
(E) If f is onto then h is onto.

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24. What is the total number of additions and multiplications in the following code?

```
s = 0;
for (i = 1; i <= n; ++i) {
    s = s + i;
    for (j = 1; j <= i; ++j) {
        s = s + j*i;
    }
}
s = s + 10;
```

(A) n;
(B) n^2;
(C) n^2+2n;
(D) $n(n+1)$;
(E) $(n+1)^2$.

25. Let a, b, m be positive integers. Which of the following is false?

(A) If $a \equiv b \mod m$, then $2a \equiv 2b \mod m$;
(B) If $a \equiv b \mod 2m$, then $a \equiv b \mod m$;
(C) If $a \equiv b \mod m$, then $2a \equiv 2b \mod 2m$;
(D) If $a \equiv b \mod m^2$, then $a \equiv b \mod m$;
(E) If $a \equiv b \mod m$, then $a \equiv b \mod 2m$.

26. How many positive divisors does $2^5 \cdot 5^8 \cdot 7^3$ have?
(A) $5 \cdot 8 \cdot 3$ (B) $4 \cdot 7 \cdot 2$ (C) $6 \cdot 9 \cdot 4$ (D) $5 \cdot 8 \cdot 3 \cdot 2$ (E) None of the above.

27. Let S be the set of all sequences of 0's, 1's and 2's of length 10. For example, S contains 0211012201. Which of the following is false?
(A) There are 3^{10} elements in S;
(B) There are $\binom{10}{5}$ sequences in S having exactly five 0's and five 1's;
(C) There are $\frac{10!}{3!4!3!}$ sequences in S having exactly three 0's, four 1's, and three 2's;
(D) There are $\binom{10}{3}2^7$ sequences in S having exactly three 0's.
(E) None of the above.

28. Let D = {a, b, c, d, e} and let L = {1, 2, 3, 4}. Which of the following is true?
(A) There are 4^5 functions from the set D to the set L;
(B) There are 5^4 functions from the set D to the set L;
(C) There are $4^5 - 12 \cdot 3^5 + 4 \cdot 2^5 - 4$ functions that map D onto L;
(D) There are $5^4 - 5 \cdot 4^4 + 4$ functions that map D onto L;
(E) None of the above.

29. How many integers between 1 and 1000 are not divisible by 2, 5, or 17?
(A) 376 (B) 377 (C) 378 (D) 379 (E) None of the above

30. A 100-element set S has subsets A, B, and C of sizes 50, 70, and 65, respectively. Which of the following statement is false?
(A) The smallest $|A \cup B \cup C|$ could be 70; (B) The largest $|A \cup B \cup C|$ could be 100;
(C) The smallest $|B \cap C|$ is 35; (D) The smallest $|A \cap B \cap C|$ could be 0;
(E) The smallest $|(A \cap B) \cup (A \cap C)|$ could be 0.

31. How many ways are there to break a 10-element set of indistinguishable objects into 3 disjoint nonempty subsets? (A) 6 (B) 8 (C) 9 (D) 10 (E) 12

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32. Consider a complete graph of n vertices, $n \geq 4$. Which of the following is false?
(A) The number of paths of length 3 is n(n-1)(n-1)(n-1);
(B) The number of paths of length 3 whose vertex sequence consist of distinct vertices is n(n-1)(n-2)(n-3);
(C) The number of paths of length 2 consisting of distinct edges is n(n-1)(n-2);
(D) The number of paths of length 3 consisting of distinct edges is n(n-1)(n-2)(n-2);
(E) None of the above.

33. A and B are matrices. Which of the following is false?
(A) $A = (A^T)^T$ for all A;　(B) If $A^T = B^T$, then A = B;　(C) If $A = A^T$, then A is a square matrix;
(D) If A and B are the same size, then $(A+B)^T = A^T + B^T$;　(E) None of the above.

34. Let A and B be 3x3 matrices defined by A[i,j]=ij and B[i,j]=i+j^2. Which of the following is false?
(A) A+B=i+ij+j^2;　(B) $\sum_1^3$ A[i,i] = 14;　(C) $A = A^T$;　(D) $B = B^T$;　(E) None of the above.

35. Define the equivalence relation R on **Z** (**Z**: integers) by (m, n) ∈ R if m ≡ n (mod 8).
How many equivalence classes of R are there?
(A) 6　(B) 7　(C) 8　(D) 9　(E) 10

36. What is the smallest equivalence relation on {1, 2, 3}?
(A) {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)};
(B) {(1,1), (2,2), (3,3)};
(C) {(1,1), (1,2), (2,1), (2,2), (3,3)};
(D) {(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,3)};
(E) {(1,1), (1,3), (2,2), (3,1), (3,3)}.

37. Let R be the relation defined on N x N (N: natural numbers) as follows: ((m, n), (p, q)) ∈ R if and only if m ≡ p (mod 3) or n ≡ q (mod 5). Which of the following is false?
(A) R is an equivalence relation;　(B) R is reflexive;　(C) R is symmetric;
(D) R is not transitive;　(E) None of the above.

38. Which of the following relation R is not an equivalence relation?
(A) For m, n ∈ N (N: natural numbers), define (m, n) ∈ R if m + n is an odd integer;
(B) Let V be the set of vertices of a graph G, and for u, v ∈ V define (u, v) ∈ R if u = v or there exists an edge from u to v;
(C) Let V be the set of vertices of a digraph D, and for u, v ∈ V define (u, v) ∈ R if u = v or there exists a path from u to v;
(D) Let Σ be some alphabet and let Σ* consist of all strings using letters from Σ. For strings s and t in Σ*, we define (s, t) ∈ R provided that length(s) ≤ length(t);
(E) None of the above.

39. Let a binary relation defined as R = {(0, 0), (1, 1)} on A = {0, 1, 2, 3}. Which of the following is true?
(A) reflexive, not symmetric, transitive;　(B) not reflexive, symmetric, transitive;
(C) reflexive, symmetric, not transitive;　(D) reflexive, not symmetric, not transitive;
(E) reflexive, symmetric, transitive.

40. Define a binary relation R on a set X to be antireflexive if xRx does not hold for any x ∈ X. The number of symmetric, antireflexive binary relations on a set of ten elements is:
(A) 2^{10};　(B) 2^{45};　(C) 2^{50};　(D) 2^{55};　(E) 2^{90}.

41. Let R and S be binary relations on a set X. Suppose that R is reflexive, symmetric, and transitive and that S is symmetric, and transitive but is not reflexive. Which statement is true for any such R and S?
(A) R ∪ S is symmetric but not reflexive and not transitive;
(B) R ∪ S is symmetric but not reflexive;
(C) R ∪ S is transitive and symmetric but not reflexive;
(D) R ∪ S is reflexive and symmetric;
(E) R ∪ S is symmetric but not transitive.

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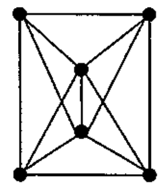
42. Let R = {(a, a), (a, b), (b, b), (a, c), (c, c)} be a partial order relation on Σ = {a, b, c}. Let $\leq$ be the corresponding lexicographic order on Σ^*. Which of the following is true?
(A) bc $\leq$ ba; (B) abbaaacc $\leq$ abbaab; (C) abbac $\leq$ abb;
(D) abbac $\leq$ abbab; (E) abbac $\leq$ abbaac.

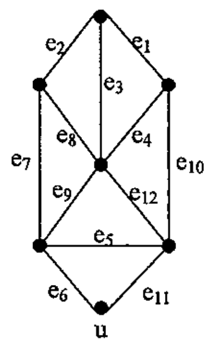
43. Consider the graph G shown on the right. Which of the following is true?
(A) G is a complete graph and a bipartite graph;
(B) G is a complete graph but not a bipartite graph;
(C) G is not a complete graph but a bipartite graph;
(D) G is not a complete graph, nor a bipartite graph.



44. Consider again the graph G shown on the right. Which of the following is true?
(A) G has an Euler circuit and a Hamilton circuit;
(B) G has an Euler circuit but no Hamilton circuit;
(C) G has no Euler circuit but a Hamilton circuit;
(D) G has neither an Euler circuit, nor a Hamilton circuit.

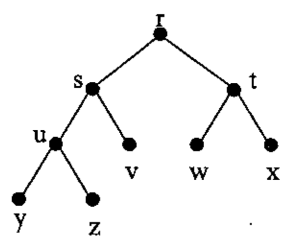
45. Consider a full binary tree with n vertices and t leaves. Let h be the height of the rooted tree. Which of the following is false?
(A) If a vertex is at level 5, then there is a path of length 5 from it to the root;
(B) h $\geq \log_2 t$; (C) h > $\log_2$n-1; (D) If n $\geq$ 3, then there are two vertices at level 1;
(E) None of the above.

46. Consider the weighted graph G on the right, where $W(e_1) < W(e_2) < \ldots < W(e_{11}) < W(e_{12})$. Which of the following shows the list of edges in the order selected by the Kruskal's algorithm?
(A) $e_1, e_2, e_3, e_5, e_6, e_7$; (B) $e_1, e_2, e_4, e_5, e_6, e_7$;
(C) $e_1, e_2, e_3, e_5, e_6, e_8$; (D) $e_1, e_2, e_3, e_4, e_5, e_6$;
(E) $e_1, e_2, e_3, e_4, e_7, e_8$.



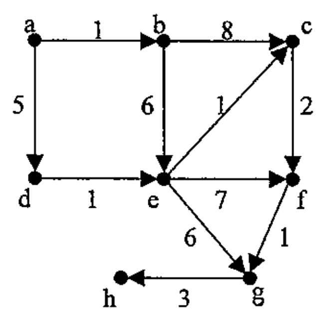
47. Consider again the weighted graph G on the right, where $W(e_1) < W(e_2) < \ldots < W(e_{11}) < W(e_{12})$. Which of the following shows the list of edges in the order selected by the Prim's algorithm, starting at vertex u?
(A) $e_6, e_5, e_7, e_2, e_1, e_3$; (B) $e_6, e_{11}, e_5, e_7, e_9, e_2$;
(C) e_6, e_5, e_9, e_7, e_2; (D) $e_1, e_2, e_3, e_4, e_5, e_6$;
(E) $e_{11}, e_6, e_5, e_7, e_9, e_2$.

48. Consider complete bipartite graph Km,n. Which of the following is false?
(A) K2,7 has an Euler path; (B) K4,6 has an Euler path;
(C) K4,5 has an Euler path; (D) K4,5 has a Hamilton path;
(E) K4,4 has an Hamilton path.



49. Consider the tree T on the right. Which of the following is false?
(A) T is a binary tree;
(B) T is a regular tree;
(C) The postorder listing of the vertices of the tree is yzuvswxtr;
(D) The inorder listing of the vertices of the tree is rstuvwxyz;
(E) The height of the tree is 3.

50. Consider the digraph D on the right. Which of the following is false?
(A) The weight of the minimal path from a to h is 13;
(B) The weight of the minimal path from b to h is 14;
(C) The weight of the minimal path from c to h is 6;
(D) The weight of the minimal path from d to h is 8;
(E) None of the above.



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